PX436: General Relativity

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Contents

0.1 Introduction

Welcome to my notes for the Warwick module PX436: General Relativity. This course is split into two halves: one to setup the mathematical framework and derive the Einstein Field Equations; the other to explore particular solutions and their applications. These notes aim to cover the lecture content as it was taught in the 2024-25 academic year whilst supplanting parts where I thought it could deal with more information. Thus the first half of these notes will be more mathematical than the lectures and problem sheets suggest. All of the pure maths mentioned in these notes but not the lectures are **non-examinable** and there for clarity.

0.1.1 How to use the guide

Anything in a white box with a blue title frame like

Title here

Unlike in previous guides, where this blue box is for memorisation, for these notes, these just exemplify main equations or points.

Any equation which contains a regular, black box like $\boxed{\textit{this}}$ is important information but you shouldn't need to memorise it.

This guide is very detailed, almost like its own set of lecture notes. It aims to answer as many questions as possible regarding both the maths and the physics in this module. Any parts non-examinable will be explicitly marked non-examinable. Beware that this can change over the years if this guide isn't updated and therefore check with the lecturer.

0.1.2 Tips

- It is guaranteed that at least one thing from every chapter will be examined, though there is a bias towards the applications
- Do **all** the problem sheets and past papers - they are very good and many appeared on the 2024 exam paper.
- Think a lot about frames of reference and what you're aiming to transform to.

0.1.3 Credits

First, thank you to those who submitted feedback

- Roy S
- Sam D
- Ladislas W

A big thank you to

- Prof. Tony Arber, whose well-delivered General Relativity lectures I had the pleasure of attending, and which these notes are derived from.
- Dr. Gareth Alexander who has a brilliant set of typeset notes for an older version of this module.

Chapter 1

Einstein Field Equations

The following sections will concentrate on the formalism and derivation of the Einstein Field Equations. The beginning of this chapter is rearranged differently from the lectures, with all the maths and general equations reshuffled to the beginning so it will be dense. Then after that, the energy-momentum tensor is defined and the rest of the course follows as in the lectures.

1.1 Why GR? Issues with Newton (his theory, not himself)

Special relativity - classical physics of flat space-time with **no gravity**. General relativity introduces **curvature** into space-time and **gravity**. GR attempted to solve some issues with Newtonian gravity:

- Newton incorrectly predicted the precession of the perihelion (point of closest approach to the Sun) to be $\simeq 5557$ are-seconds/century, due to the Sun and planets with Jupiter having the largest impact of the planets
- A correction posted in 1846 that led to Uranus to be found
- Einstein would correct this correction *by* 43 arcseconds per century (still not as slow as my brain).

1.2 Equivalence Principle

Equivalence Principle

Gravitational mass m_g = Inertial mass m_I OR **Locally**, gravity and acceleration are indistinguishable.

1.2.1 Einstein Thought Experiments

To affirm the second definition of the equivalence principle, Einstein devised some thought experiments, and as physicists, we like thinking. Consider 3 sealed boxes with no external forces, so no weak, strong, EM etc. They are defined in the following Figure 1.1: Notice,

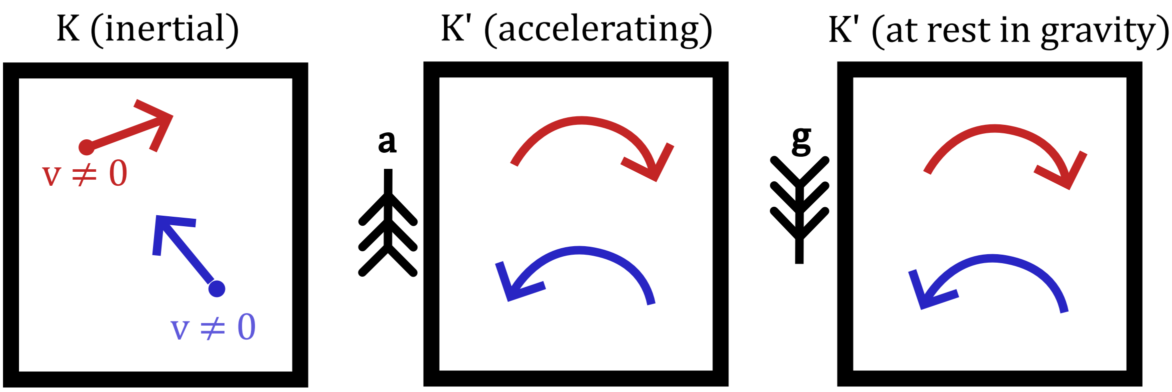


Figure 1.1: Caption

accelerating the room upwards causes the paths to curve (imagine throwing a ball straight ahead, then moving your eyes upwards - to you, the path curves downwards). This is locally equivalent to the third room, where there is no assumed acceleration but gravity is acting (same ball scenario, but this time throw the ball and keep eyes fixed). The latter two rooms are only **locally equivalent** since if you consider a same fixed point in both rooms, you wouldn't be able to tell which scenario is happening until you looked at a global scale.

Another thought experiment is the **deflection of light**. We take the same boxes as Fig. 1.1, but this time with a single horizontal (left-to-right) photon. Equivalence principle tells us, the same thing happens, where light will bend downwards in this case. Although the photon has zero mass, we can ‘cancel them out’ (don’t worry about dividing by zero here just don’t think about that when thinking about this).

Finally, we have probably the most innocuous example: a rotating disk. Let the reference frame K be a disk of diameter D and circumference C. Obviously, $\pi = C/D$. Now let the frame K' be one where you are constantly rotating around the disk. Weirdly enough, the new ratio $C'/D' > \pi$! Note that $D = D'$ because the diameter is perpendicular to the motion, so SR tells us its length is unchanged. The conclusion of the circumference-to-diameter ratio being greater than π can be argued as so:

1. Break the circumference down into infinitesimal ”straight bits”, or think about many infinitesimal rulers
2. SR $\implies$ length contraction of each bit (ruler)
3. So more of these bits (rulers) are needed to complete the circle in K'
4. So $C' \neq C$.

Since rotation implies a central acceleration, this implies a central gravity, so by the equivalence principle, the space around the disk is curved!
Remark. The reason for this is because in differential geometry (RIP MA4C0 Differential Geometry takers), GR is casted in hyperbolic space - here, a regular flat circle in flat space-time keeps the same radius but an increased circumference in hyperbolic space, as the angular motion of the disk causes distorts the manifold. You’ve also got to worry about synchronising time when solving this problem.

1.2.2 Field Equation

The Newtonian field equation is a type of Poisson partial differential equation

$$\nabla^2 \phi = 4\pi G \rho, \tag{1.1}$$

where ϕ is some scalar potential and ρ a mass density. Effectively, ∇^2 relates to curvature, ρ to energy density and ϕ to spacetime geometry. This will get us to the Einstein Field Equations (spoilers!).

1.3 Differential Geometry

A bit of maths without the formalism of a maths course. These definitions and statements are formally **non-examinable** but help to explain some jumps in derivations. These definitions (slightly modified) are taken from Maxwell Stolarski’s notes for MA4C0 Differential Geometry. One of the most important visualisations you can make for a manifold is take some real surface (e.g., a blanket) and imagine you are fixed to the surface of it. You can only move along the surface, you cannot go away from it. This means if you are at a point p, whatever direction you will move will always be a **tangent vector** (whatever this precisely means) to p and so we can only talk about manifolds and GR in terms of tangent vectors and local spaces as we will see.

You might wonder then, ‘but real space-time is not a blanket’ - there are 3 spatial dimensions. Unlike a blanket, where we could *embed* it into our 3D interpretation of space, we are three-dimensional beings that live in our own three-dimensional manifold, so we can’t simply shoot ourselves somewhere and start drawing tangent planes easily. But imagine for a moment, a fourth spatial dimension: if you shoot yourself into space somewhere to a point p, the same idea applies. No matter where you choose to go, there will be a tangent vector in that direction. However, the non-linearity and locality of manifolds (since if you move to point q, the tangent space may necessarily not be the same as in p) requires careful treatment of quantities like acceleration whilst also ensuring the same relativity principles. Of course, we don't have a fourth spatial dimension, so rigorous treatment of the changes of vector fields and so on is necessary. This is one of the key ideas behind GR and why Einstein used differential geometry.

Definition 1.3.1. The notation $C^\infty(X)$ on a set X denotes an infinitely-differentiable (the C^∞-part) function on $X \to \mathbb{R}$. The map may be notated more explicitly.

1.3.1 Manifolds (non-examinable)

Definition 1.3.2. A **smooth manifold** M is a space where

- Given any (open) subset U on M, it can be written as union of a countable open basis set $\mathcal{U} = \cup_{\alpha\in\mathcal{A}} U_\alpha$
- It has a bijective, continuous, infinitely-differentiable map $\phi_\alpha : U \to \mathbb{R}^n$, called a **smooth chart**, whose inverse is also continuous) ϕ_α for every U_α

Moreover, the collection $\{(U_\alpha, \phi_\alpha)\}$ is a **smooth atlas of charts**. This can be thought of as a set which describes regions of the manifold and only works on **local regions**. This is why we strictly we have a collection of open sets since we need to break down curved space into 'local regions of Euclidean space'.

Definition 1.3.3. Let M be a smooth manifold, and some point $p \in M$. Then the **tangent space** T_pM is defined as:

$$T_pM = \mathbb{R} \cdot \left\{ \left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p \right\} \tag{1.2}$$

where $(x^1, \ldots, x^n)$ are the **local coordinates**, i.e. the basis that makes up the local Euclidean space around p.

Questions you may have:

- What do the derivatives act on? Well, they act on functions $M \to \mathbb{R}$. Easy example: suppose $M = \mathbb{R}^2$. The derivatives are $\frac{\partial}{\partial x^1}$ and $\frac{\partial}{\partial x^2}$. Therefore the tangent plane (line in this case) will be the tangent to any point $p = (x, y) = (x, f(x))$ that lies in graph on the plane.
- the notation $\mathbb{R}\cdot$ just means that its the real number system projected onto the set of derivatives, because that is what the tangent space consists of.

Intuitively, T_pM can be thought of as every vector which can be tangent to a point p. E.g. take some surface in $\mathbb{R}^3$. At every point p, you can locally fix a Euclidean system (x^1, x^2, x^3). Then every tangent vector is going to be a derivative projected in each direction of x^i.

Note that the local coordinates you can define around p is not unique, e.g. you can rotate the coordinate system. This is done by a change of basis $z^i = A^{ij}x^j$:

$$\left.\frac{\partial}{\partial x^i}\right|_p = \sum_{j=1}^n \left.\frac{\partial \tilde{x}^j}{\partial x^i}\right|_p \left.\frac{\partial}{\partial \tilde{x}^j}\right|_p = \left.\frac{\partial \tilde{x}^j}{\partial x^i}\right|_p \left.\frac{\partial}{\partial \tilde{x}^j}\right|_p \tag{1.3}$$

using Einstein summation convention over j in the last equality.

Definition 1.3.4. Tangent vectors $X = c^i \frac{\partial}{\partial x^i}\big|_p \in T_pM$ act on smooth functions $f : M \to \mathbb{R}\,(f \in C^\infty(M) = C^\infty(M \to \mathbb{R}))$ via

$$X \cdot f = df_p(X) = c^i \frac{\partial f}{\partial x^i}\bigg|_p \in \mathbb{R}$$

This solidif Using the tangent space, its **dual space** can be defined

Definition 1.3.5. The **cotangent space** to M of p is defined as

$$T_p^*M = \mathbb{R} \cdot \{dx_p^1, \dots, dx_p^n\} \tag{1.4}$$

where $dx_p^i : T_pM \to \mathbb{R}$ are linear functionals satisfying $dx_p^i \left(\frac{\partial}{\partial x^j}\big|_p \right) = \delta_j^i$.

If you stare at this hard enough, you will realise it basically evaluates to making sure that you have the correct dx^i associated with the correct $\frac{\partial}{\partial x^i}$. The dx_p^i also obey similar transformation laws:

$$dx_p^i = \frac{\partial x^i}{\partial \tilde{x}^j}\bigg|_p d\tilde{x}_p^j \quad (\forall i), \tag{1.5}$$

Now, we can consider **bundles**, where the above definitions are defined for every $p \in M$:

Definition 1.3.6. Define

$$TM := \bigsqcup_{p \in M} T_pM \qquad\qquad T^*M := \bigsqcup_{p \in M} T_p^*M \tag{1.6}$$

where TM is termed the **tangent bundle** and T^*M the **cotangent bundle**. Notation: $(p, v) \in TM$ where $p \in M, v \in T_pM$ and likewise for T^*M. Additionally,

$$\bigsqcup_{\alpha \in \mathcal{A}} A_\alpha = \bigcup_{a \in \mathcal{A}} \{(\alpha, x) : x \in A_\alpha\}$$

is the **disjoint union**.

Both of these bundles are themselves smooth manifolds (non-examinable).

Definition 1.3.7. A **vector field** is a smooth map $\xi : M \to TM$ such that

$$\xi(p) = \xi_p \in T_pM \forall p \in M \tag{1.7}$$

Namely, a vector field is a smooth choice of assigning tangent vectors to every point $p \in M$. Hence, you can think of TM as every possible vector field.

Definition 1.3.8. A **1-form** is a smooth map $\omega : M \to T^*M$ such that $\omega_p = \omega(p) \in T_p^*M \forall p \in M$.

In the lectures, 1-forms and vectors are defined in a physical sense in Week 2. In $\mathbb{R}$, the 1-form is simply $f(x)dx$, the regular 'infinitesimal area'. We shall see the earlier ramble of maths will realise itself in SR, and later GR. For now, if you're interested, here's some more definitions.

'Smooth functions between manifolds induce maps on their (co)tangent bundles.' (Maxwell Stolarski).

Definition 1.3.9. Let M, N be smooth manifolds and $f : M \to N$ is smooth. Then the **differential** of f, denoted $f^* := df : TM \to TN$ is defined as $f^*X = df_p(X) \in T_{f(p)}N$

$$df_p\left(X = X^j \left.\frac{\partial}{\partial x^j}\right|_p\right) = X^j \left.\frac{\partial f^\alpha}{\partial x^j}\right|_p \left.\frac{\partial}{\partial y^\alpha}\right|_{f(p)} \tag{1.8}$$

where $X \in T_pM$ is a tangent vector as in Definition 1.3.4. Here, $(x^i), (y^\alpha)$ are local coordinates of M, N respectively. The physics behind this definition can be elucidated: T_pM can be thought of as all possible velocities at $p \in M$, with TM every possible set of velocities for every $p \in M$. Since f maps points in M to points in N, we need to include the change of $f(p) \in N$ via the chain rule, which then just ends up being the velocities (tangent vectors) to $f(p)$. Hence the differential tells you how f changes the tangent vectors in T_pM. You will notice the derivatives on the LHS& RHS doesn't appear to be acting on anything, but it will act on a function $g : N \to \mathbb{R}$ which is smooth, and would be some quantity of interest. This will be important later as it will ensure that we can 'differentiate' things.

Similarly, the **pull-back** is defined as

$$f_* : T^*_{f(p)}N \to T^*_pM \quad (f_*\omega)_p\,(X) = \omega\,(df_p(X))$$

In local coordinates as above,

$$f_*\left(\omega_\alpha dy^\alpha_{f(p)}\right) = \omega_\alpha \left.\frac{\partial f^\alpha}{\partial x^i}\right|_p dx^i_p$$

The cotangent space is the space of linear functions ω_α acting on tangent vectors. Thus f_* effectively measures the distance travelled on M after travelling some distance on N.

1.3.2 Tensors

Every physicist knows a tensor transforms as a tensor and that's that. In this context, we will think of tensors as multilinear maps on a finite-dimensional vector space V and its dual space V^*:
Definition 1.3.10. A **contravariant** k-tensor is a multilinear map

$$F : V \times \ldots_{k \text{ times}} \times V \to \mathbb{R}$$

A **covariant** l-tensor is a multilinear map (i.e. linear in each entry)

$$F : V^* \times \ldots_{l \text{ times}} \times V^* \to \mathbb{R}$$

A $\binom{k}{l}$ tensor is a multilinear map

$$F : (V^* \times \ldots_{k \text{ times}} \times V^*) \times (V \times \ldots_{l \text{ times}} \times V) \to \mathbb{R}$$

where $V^* = \{\omega : V \to \mathbb{R} \mid \omega \text{ linear}\}$ is the space of linear functionals.

So here, the 'dimensions' of the tensor matters and this affects the notation (including raising and lowering etc.).
Definition 1.3.11. We define

- $T^k(V)$ the set of contravariant k-tensors

- $T_l(V)$ the set of covariant l-tensors
- $T^k_l(V)$ the set of $\binom{k}{l}$-tensors

Proposition 1.3.1. *Rank-1 tensors can satisfy:*

- $T^0_1(V) = T_1(V) = V^*$ *- namely that covariant 1-tensors are linear functionals, aka they are 1-forms*
- $T^1_0(V) = T^1(V) = V$ *- namely that contravariant 1-tensors are vectors!*

This follows directly from the definitions.

$$v^\mu \text{ is a vector} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{tensor} \tag{1.9}$$

$$v_\mu \text{ is a 1-form} \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{tensor} \tag{1.10}$$

So reducing the complicated diff geo definitions from earlier into something tangible:

- A **1-form** is a linear function which takes a vector and returns a scalar.
- A **vector** is a linear function which takes a 1-form and returns a scalar.
- So a $\binom{k}{l}$-tensor takes k 1-forms and l vectors to map into a scalar.

Remark. It is important to see how this comes out from the nonsense of differential geometry. From Definition 1.3.8, a 1-form is an infinitely differentiable map from a point in the manifold to the cotangent bundle. But the cotangent bundle is a union of maps to $\mathbb{R}$, so a 1-form does map vectors to scalars! Similarly from Definition 1.3.7 for vectors, a vector field maps from M into a particular tangent space, consisting of (tangent) vectors, which act on smooth real functions (Definition 1.3.4) such as 1-forms (Definition 1.3.9), so indeed vectors map 1-forms to real numbers.

In previous courses of physics, you have accepted how to 'multiply' tensors, e.g.

$$F^{\alpha\beta\gamma}_{\delta\sigma} G^{\mu\nu}_{\eta} := H^{\alpha\beta\gamma\mu\nu}_{\delta\sigma\eta}$$

This definition is formalised as the infamous **tensor product**

Definition 1.3.12. If $F \in T^k_l(V)$ and $G \in T^p_q(V)$, then the tensor product $F \otimes G$ is

$$\begin{gathered} F \otimes G : (V^*)^{l+q} \times V^{k+p} \to \mathbb{R} \\ (F \otimes G)(\omega_1, \dots, \omega_{l+q}, v_1, \dots, v_{k+p}) := \\ F(\omega_1, \dots, \omega_l, v_1, \dots, v_k)\, G(\omega_{l+1}, \dots, \omega_{l+q}, v_{k+1}, \dots, v_{k+p}). \end{gathered}$$

This forms a $\binom{k+p}{l+q}$ tensor.

Definition 1.3.13. A $\binom{k}{l}$**-tensor field** on a manifold M assigns a tensor to every $p \in M$ - it's the vector field analogue for tensors.

We write

$$\mathcal{T}^k_l(M) = \left\{ \begin{pmatrix} k \\ l \end{pmatrix} \text{ tensor fields} \right\} \tag{1.11}$$

This is also a vector space because you can multiply tensors by scalars and keep its properties.

Remark. This definition actually makes more sense with tensor bundles, but not really needed so I have skipped over it.

$$\begin{aligned}\mathcal{T}_1(M) &= \{\text{1-forms on } M\} \\ \mathcal{T}^1(M) &= \{\text{ vector fields on } M\}\end{aligned} \tag{1.12}$$

Additionally, it is possible to define a map over k-copies of $\mathcal{T}^1(M)$ and l copies of $\mathcal{T}_1(M)$ mapping to the space of smooth functions (on vector fields or 1-forms) which is linear in every entry, namely

Lemma 1.3.1. *(Tensor Characterization Lemma). A map*

$$\tau : \mathcal{T}^1(M) \times \cdots_{k \text{ times}} \times \mathcal{T}^1(M) \times \mathcal{T}l1(M) \times \cdots_{l \text{ times}} \times \mathcal{T}l1(M) \to C^\infty(M)$$

where

$$\tau(\ldots, fX + gY, \ldots) = f\tau(\ldots, X, \ldots) + g\tau(\ldots, Y, \ldots)$$

for $f, g \in C^\infty(M)$, the space of smooth functions.

τ comes from a $\binom{k}{l}$ tensor field $F \in \mathcal{T}_l^k(M)$ as above if and only if τ is $C^\infty(M)$-linear in each entry.

Why is this important? Later, we will see objects that *are* multilinear maps but are not tensors in the way we have defined them above. This is because they have slightly different transformation rules because of their construction. The main example are Christoffel symbols, which by themselves are *not* tensors. However we can apply operations such as the gradient and multiplying Christoffel symbols together, to make a tensor (Riemann curvature tensor).

1.3.3 Metrics

GR is all about finding the right metric (see Chapter 2) for different situations. However it is important to analyse metrics, what they are, how they work and their link to the manifolds and tensors above. In SR and GR, we work with a **pseudo-Riemannian metric**.

Definition 1.3.14. Let M be a manifold. A (fully-covariant) **pseudo-Riemannian metric** g is a $\binom{2}{0}$-tensor field $g \in \mathcal{T}_0^2(M)$ which is symmetric: $g_p(v, w) = g_p(w, v)$ $\forall p \in M$ and $v, w \in T_pM$.

The pair (M, g) is a **pseudo-Riemannian manifold**.

In GR, we use index notation, so we often write it as $g_{\alpha\beta}$.

The key example is the Minkowski metric $\eta_{\mu\nu}$ defined in SR. Written in matrix form:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{1.13}$$

where we are using $(-+++)$ signature for the rest of this course. This is indeed symmetric.

It is a fact that not every smooth manifold can have a pseudo-Riemannian metric, which is why half the problem in GR is finding a suitable metric - it must obey physical constraints and observations WHILST also not breaking maths.

Metrics also satisfy a very neat relationship, which is worth knowing

Proposition 1.3.2. *The metric $g_{\alpha\beta}$ has a well-defined inverse $g^{\alpha\beta}$ and satisfies*

$$g^{\nu\gamma} g_{\gamma\beta} = \delta^\nu_\beta \tag{1.14}$$

Raising and Lowering operators

We've been doing this all the time in physics, time to see why. As always, let (M, g) be a pseudo-Riemannian manifold.

Definition 1.3.15. The **lowering map**

$$\mathrm{L} : TM \to T^*M \qquad X \mapsto L(X) := g(X, \bullet), \tag{1.15}$$

where the bullet is an element of another tangent space (recall TM is the union of tangent spaces which is all vectors tangent to a point $p \in M$). In local coordinates,

$$L(X) = g\left(X^i \frac{\partial}{\partial x^i}, \bullet\right) = g_{kj} dx^k \left(X^i \frac{\partial}{\partial x^i}\right) dx^j(\bullet) = g_{kj} X^i \delta_i^k dx^j = g_{ij} X^i dx^j$$

As physicists, this notational garbage really boils down to

$$L(X) = X_j dx^j = g_{ij} X^i dx^j \tag{1.16}$$

so we really see that

$$\boxed{\text{lowering an index is the same as multiplying by the fully-covariant metric tensor.}}$$

Definition 1.3.16. The **raising map**

$$\mathrm{R} : T^*M \to TM \tag{1.17}$$

In local coordinates,

$$\omega = \omega_i dx^i \mapsto R(\omega) := g^{ij} \omega_i \frac{\partial}{\partial x^j}$$

where we write $\omega^i := g^{ij} \omega_j$.

$$\boxed{\text{raising an index is multiplying a 1-form by a fully-contravariant metric tensor.}}$$

Coordinate Transforms

The last part of maths needed for this course is coordinate transforms and finding the metric. Suppose we can accurately define a position vector (from some reference origin) like in Minkowski flat space-time $\underline{X}(x^1, x^2, \ldots)$. We can find the displacement vector as

$$d\underline{X} = dx^i \frac{d\underline{X}}{dx^i}$$

Now take 2 arbitrary directions (could be curvy like in Fig. 1.2) - we can do the same calculation:

$$|d\underline{X}|^2 = \binom{2}{\min(i,j)} (\partial_{x^i} \underline{X}) \cdot (\partial_{x^j} \underline{X}) dx^i dx^j = g_{ij} dx^i dx^j$$

Specifically,

$$\text{Define } \frac{\partial \underline{X}}{\partial x^i} := \underline{e}_i \qquad g_{ij} = \underline{e}_i \cdot \underline{e}_j \tag{1.18}$$

and in GR, this generalises to $g_{\mu\nu} = \underline{e}_\mu \cdot \underline{e}_\nu$

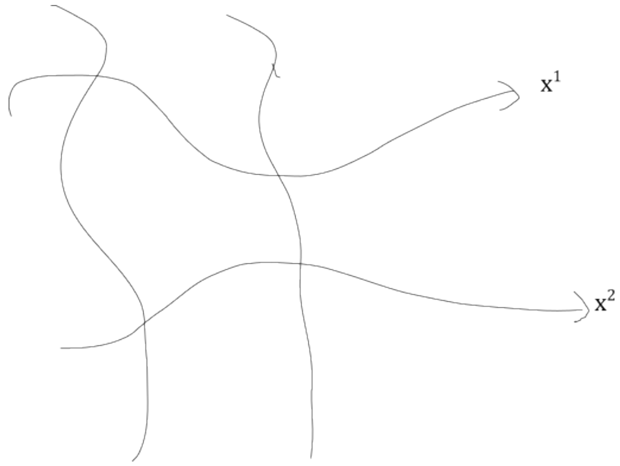


Figure 1.2: Arbitrary 2D coordinates.

Let's take polar coordinates as an example. In the Cartesian plane, any position vector $\mathbf{R} = x\hat{\imath} + y\hat{\jmath}$ with $\underline{i, j}$ the regular unit vectors for x, y respectively. We suppose this vector has length $r = \sqrt{x^2 + y^2}$. In polar coordinates, this can be written as a radial vector out of length r, oriented at some angle θ, so $\mathbf{R} = r\underline{e}_r$. In this case, $\underline{e}_r = \hat{r}$ is the radial unit vector. But we know there is a second vector $\underline{e}_2$ which we can find, because originally there were 2 basis vectors $\hat{\imath}, \hat{\jmath}$, and by some linear algebra, we know the total number of basis vectors must be the same for any vector space, even if the basis is not unique. Following a similar procedure to above,

$$\underline{e}_2 = \frac{\partial \underline{R}}{\partial x^2} = \frac{\partial R}{\partial \theta} = r\frac{d}{d\theta}\underline{e}_r = r\frac{d}{d\theta}\underline{r} = r\hat{\theta} = \underline{e}_\theta \tag{1.19}$$

This new coordinate vector is **not a unit vector**. In any case, we can use the definition of g_{ij} and find:

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \tag{1.20}$$

and the interval becomes $ds^2 = dr^2 + r^2 d\theta^2$

1.4 Special Relativity

We've done this before, but now write it in notation that is ready for GR. Consider 2 frames S and S' where S' moves at speed $\beta = v/c$ relative to S, along the x direction. The standard Lorentz Transforms (LT) apply:

$$x' = \gamma(x - vt) \qquad t' = \gamma(t - xv/c^2) \qquad \gamma = (1-\beta)^{1/2}, \tag{1.21}$$

and of course, $y = y', z = z'$ in this scenario. We can define $(x^0, x^1, x^2, x^3) = (ct, x, y, z)$. Then

$$x^{\mu'} = \sum_{\nu=0}^{3} \Lambda^{\mu'}_{\nu} x^\nu = \Lambda^{\mu'}_{\nu} x^\nu, \tag{1.22}$$

where

$$\Lambda^{\mu'}_{\nu} x^\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.23}$$

Lemma 1.4.1.

$$\Lambda^{\mu}_{\nu}\Lambda^{\nu}_{\sigma} = \delta^{\mu}_{\sigma} = \begin{cases} 1 & \mu = \sigma \\ 0 & \mu \neq \sigma \end{cases} \tag{1.24}$$

Definition 1.4.1. The **interval** is defined as

$$s^2 = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 \tag{1.25}$$

and is invariant under LT

However, this assumes that the reference frame has a well-defined origin to measure x^{μ} from. We will soon see this is basically impossible in curved space.
Definition 1.4.2. The **interval** (oof) for differential changes is defined as

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \tag{1.26}$$

and is also an invariant scalar.
Remark. This module uses the $(-+++)$ signature whereas other sources may use $(+---)$. Nothing different, just causes some sign changes in equations.
Definition 1.4.3. The contravariant **4-position** is defined as

$$x^{\mu} = (ct, x, y, z) \tag{1.27}$$

We can lower the index and get the covariant 4-position,

$$x_{\mu} = (-ct, x, y, z) \tag{1.28}$$

Let τ be the proper time, which is the time elapsed in an inertial frame. Then $dt = \gamma d\tau$ and
Definition 1.4.4. The contravariant **4-velocity** is the proper time derivative of the 4-position

$$\frac{dx^{\mu}}{d\tau} = \lim_{d\tau\to 0} \frac{x^{\mu}(\tau + d\tau) - x^{\mu}(\tau)}{d\tau} = u^{\mu} = (c, v_x, v_y, v_z) \tag{1.29}$$

Note the missing γ here, since in the frame where proper time is measured, the observer is stationary in that frame and $\gamma = 1$.

We can also take gradients of 4-vectors.
Definition 1.4.5. The covariant **4-gradient** with respect to a contravariant 4-vector is defined as

$$\frac{\partial}{\partial x^{\alpha}} = \partial_{\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = (\partial_0, \partial_1, \partial_2, \partial_3) \tag{1.30}$$

The contravariant 4-gradient (w.r.t covariant terms)

$$\frac{\partial}{\partial x_{\alpha}} = \partial^{\mu} = \left(-\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = (-\partial_0, \partial_1, \partial_2, \partial_3) \tag{1.31}$$

1.4.1 Minkowski space-time

The interval can be written as

$$ds^2 = \eta_{\mu\nu} dx^{\mu} dx^{\nu} \tag{1.32}$$

where $\eta_{\mu\nu}$ is the Minkowski metric tensor with components $\eta_{00} = -1$ and $\eta_{ii} = 1$ for $i = 1, 2, 3$. All other components are 0. It is a symmetric bilinear form (whatever that means) and a type $\binom{0}{2}$-tensor. By the definition of a tensor, we see that it has 2 components and accepts two vectors to return a scalar.

In SR/Minkowski space-time, the 4-position $\underline{x} = (ct, x, y, z) = x^\mu \underline{e}_\mu$ where $\underline{e}_i$ is the basis vector whose components are 0 everywhere except the ith position, where it is 1. Indeed since Minkowski is 4D, there are 4 basis vectors. As before, we can transform reference frames using the

1.4.2 The Lorentz Group

The group (in the mathematical sense) of all Lorentz transformations forms the **Lorentz group**. In lectures, you may have seen that the Lorentz transform 'tensor' $\Lambda^{\alpha'}_\alpha$ is not a tensor - this is because it explicitly depends on the 2 references frames you are transforming between - you are describing *how* things (vectors) change. Even though in index notation, it looks like a $\binom{1}{1}$-tensor, you can't just write an expression for $\Lambda^{\alpha'}_\alpha$ that is true in *all* coordinate frames.

The Lorentz transform is actually a multilinear map, like tensors, and for Minkowski 3+1 spacetime (3 spatial, 1 time) with the $(- + ++)$ signature, it lies in a non-abelian (Lie) group with a *matrix representation*. This is related to a Problem Sheet question where you are asked to transform the Energy-Momentum Tensor from a frame where a fluid is at rest, to a frame where it is moving in the positive x-direction at speed v. Now, this question is subtle. Since the fluid is moving in the positive direction in this frame, the **frame is moving in the negative direction** $-v$, so all the $-\gamma\beta$ terms become $\gamma\beta$.

$$T^{\alpha'\beta'} = \Lambda^{\alpha'}_\alpha \Lambda^{\beta'}_\beta T^{\alpha\beta}. \tag{1.33}$$

Now, we have applied the Lorentz transform matrix twice, once for each index (since we have a $\binom{2}{0}$-tensor). You might go ahead and start evaluating, but take a step back because we are trying to do matrix multiplication here. So actually,

$$T^{\alpha'\beta'} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \rho_0 c^2 & 0 & 0 & 0 \\ 0 & p_0 & 0 & 0 \\ 0 & 0 & p_0 & 0 \\ 0 & 0 & 0 & p_0 \end{pmatrix} \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.34}$$

because $\eta = \Lambda^T \eta \Lambda$ for some matrix η. But why is this the case? Well remember that the interval Eq. (1.32) is invariant under all frames, and this is how we define a Lorentz transform between primed and unprimed coordinates:

$$x'^\alpha \eta_{\alpha\beta} x'^\beta = x^\mu \eta_{\mu\nu} x^\nu \tag{1.35}$$

Juggling some more indices,

$$\begin{aligned} x^\mu \Lambda^\alpha{}_\mu \eta_{\alpha\beta} \Lambda^\beta{}_\nu x^\nu &= x^\mu \eta_{\mu\nu} x^\nu, \\ \Lambda^\alpha{}_\mu \eta_{\alpha\beta} \Lambda^\beta_\nu &= \eta_{\mu\nu} \end{aligned} \tag{1.36}$$

Now, the α index is selecting rows, whereas β is selecting columns, the first Λ is actually the **transpose** (although in this case, Λ is symmetric so $\Lambda^T = \Lambda$).

This corresponds identically to the group of 4-dimensional orthogonal matrices which satisfy

$$R^T I_4 R = I_4 \tag{1.37}$$

where I_4 is the 4×4 identity matrix. In the case of Lorentz transforms, the metric tensor also satisfies this relationship.

1.5 Tensor Equations

So we have been doing a lot of work trying to incorporate tensors into our description of SR - why? It is always about **invariants**. Suppose we have a 4-vector x^α which transforms as follows:

$$x^\alpha \mapsto x^{\alpha'}(x^0, x^1, x^2, x^3), \tag{1.38}$$

The vectors, 1-forms and $\binom{0}{2}$ tensors transform as

$$v^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^\alpha} v^\alpha \tag{1.39}$$

$$v_{\alpha'} = \frac{\partial x^\alpha}{\partial x^{\alpha'}} v_{\alpha'} \tag{1.40}$$

$$g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu} \tag{1.41}$$

Memorising is a bit of a stretch but you should know the intuition behind these transforms.

This is because ds^2 is invariant across reference frames, i.e. each set of coordinates satisfies

$$ds^2 = g_{\mu'\nu'} dx^{\mu'} dx^{\nu'} = g_{\mu\nu} dx^\mu dx^\nu \tag{1.42}$$

Therefore using how coordinate transforms Eq. (1.5), then

$$g_{\mu'\nu'} dx^{\mu'} dx^{\nu'} = g_{\mu\nu} \frac{\partial x^\mu}{\partial x^{\mu'}} dx^\mu \frac{\partial x^\nu}{\partial x^{\nu'}} dx^\nu = g_{\mu\nu} dx^\mu dx^\nu \tag{1.43}$$

which derives those set of 3 equations. To see this more concretely, we look at further examples of coordinate transforms.

For a general $\binom{k}{l}$ tensor $F^{\mu_1,\ldots\mu_k}_{\nu_1\ldots\nu_l}$, it obeys the transformation

$$F^{\mu'_1,\ldots\mu'_k}_{\nu'_1\ldots\nu'_l} = \left[\prod_{i=1}^{l} \frac{\partial x^{\nu_i}}{\partial x^{\nu'_i}} \prod_{i=1}^{k} \frac{\partial x^{\mu'_i}}{\partial x^{\mu^i}} \right] F^{\mu_1,\ldots\mu_k}_{\nu_1\ldots\nu_l} \tag{1.44}$$

1.5.1 Curved metric

Consider the surface of a sphere as in Fig.1.3(a) Define θ, ϕ as usual polar and azimuth angles respectively. r is the measured distance *along the surface* from the (north) pole. A is a point in the sphere such that $\overrightarrow{AP} = R\sin\theta$. R is the radius of the sphere if it was measured in 3D. The interval between 2 points is clearly

$$ds^2 = dr^2 + R^2 \sin^2(\theta) d\phi^2$$

but $r = R\theta$, so we can substitute this in above and

$$ds^2 = dr^2 + R^2 \sin^2(r/R) d\phi^2$$

This is an **intrinsic** definition of ds^2 and g_{ij} since the quantities used all lie in the manifold, and there is no explicit use of the component vectors.

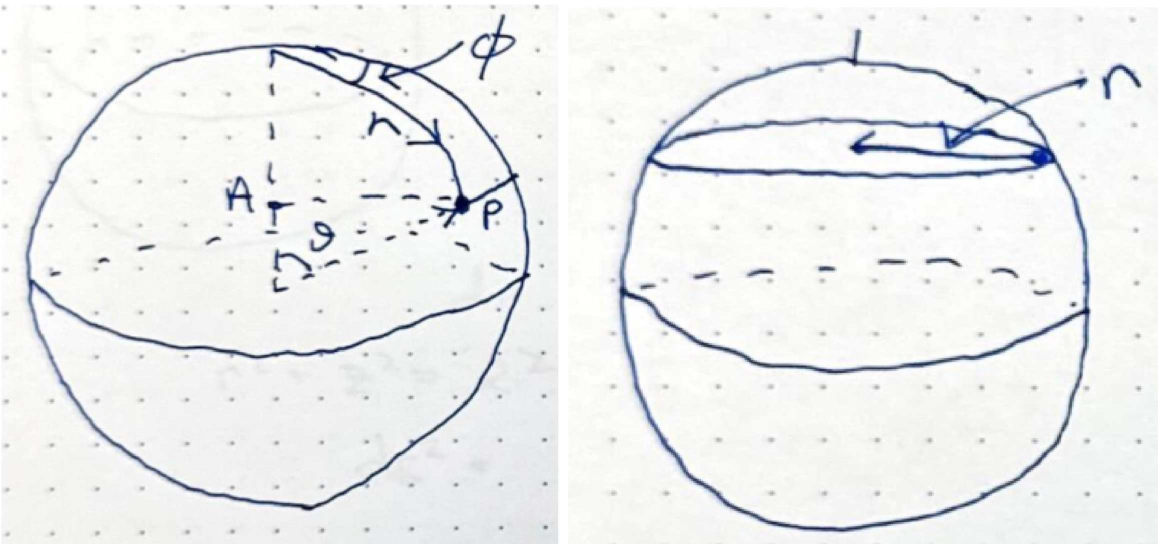


Figure 1.3: (a) General sphere manifold. (b) Same sphere but with coordinate system in the $x - y$ plane. Screenshots from Prof. Tony Arber's lectures.

1.5.2 Alternate metric

Now let's look at Fig. 1.3(b) where our r is now the radius of the circle in an $x - y$ plane. The sphere is characterised by $x^2 + y^2 + z^2 = R^2$. But by definition, $r^2 = x^2 + y^2$. Thus we can describe this sphere with a "cylindrical polar" system since we have a vertical change dz, an azimuth change $d\phi$ and an r change dr and thus

$$ds^2 = dr^2 + r^2 d\phi^2 + dz^2$$

But we know that dr, dz are not independent, since $r^2 + z^2 = R^2 \implies rdr + zdz = 0$. Substituting this into ds^2 gives

$$ds^2 = dr^2 \left(1 + \frac{r^2}{z^2}\right) + r^2 d\phi^2$$

we can now eliminate z directly with $z^2 = R^2 - r^2$

$$\begin{aligned} ds^2 &= dr^2 \left(1 + \frac{r^2}{z^2}\right) + r^2 d\phi^2 \\ 1 + \frac{r^2}{z^2} &= \frac{R^2}{R^2 - r^2} = \frac{1}{1 - \frac{r^2}{R^2}} \\ \implies ds^2 &= \frac{dr^2}{1 - \frac{r^2}{R^2}} + r^2 d\phi^2 \end{aligned}$$

- Coordinate singularity at $r = R$
- At $r = R, dr = 0$
- Coordinate singularity can be removed with $r = R \sin \chi$
- This is still a metric because it's symmetric

1.5.3 Invariance of Tensor Equations

We will see why tensor equations are so useful. Tensors involve derivatives to change coordinates. If we have a classical problem in any coordinate system,

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt}$$

then this is a valid vector/tensor equation, assuming that both sides are written in the same coordinate system. But does $F_i = mdv_i/dt$ is a valid tensor equation? NO. If we assume that $F_i = mdv_i/dt$ is true in a Cartesian plane, we cannot just swap i with say r from a polar coordinate system and expect it to be the same.

Proposition 1.5.1. *Components of the derivative of a vector are not the same as the derivative of the components of a vector, because vectors change between coordinate systems*

Proof. The proof is easy and not really the point of this module, it hasn't come up before. Set $\mathbf{v} = v_r\hat{r} + v_\theta\hat{\theta}$ and differentiate it with respect to time and find the radial component of the acceleration. You will see the expression is very much dependent on $\hat{r}, \hat{\theta}$. **We need a different type of derivative to reconcile this difference.** □

Theorem 1.5.1. *If, in a reference frame S, a tensor equation $A^{\alpha\beta} = B^{\alpha\beta}$ (whatever it may be, it could be more complicated, as long as both sides are tensors), then the **equation is true in any frame**.*

Proof. Let $A^{\alpha\beta} = B^{\alpha\beta}$ in x^μ frame. Transform to $x^{\mu'}$ then if A, B are tensors then

$$\underbrace{\frac{\partial x^\alpha}{\partial x^{\alpha'}}\frac{\partial x^\beta}{\partial x^{\beta'}}}A^{\alpha'\beta'} = \underbrace{\frac{\partial x^\alpha}{\partial x^{\alpha'}}\frac{\partial x^\beta}{\partial x^{\beta'}}}B^{\alpha'\beta'}$$

same at all points in space.

$$A^{\alpha'\beta'} = B^{\alpha'\beta'}$$

tensor equations are invariant. □

1.6 Energy-Momentum tensor $T^{\mu\nu}$

Definition 1.6.1. The **instantaneous rest frame** (IRF) is a frame in which the 3-velocity of the body is zero at that instant of time.

Suppose we have dust, modelled at zero temperature, non-interacting particles. In the instantaneous rest frame (IRF), the number density is n. In the LT frame, $n' = \gamma n$. However, Lorentz contraction occurs only in the direction of the boost. What this means is in the LT frame, the number density will look different depending on which direction we go - n' is effectively a vector. Therefore, we construct a 4-vector:

$$N^\alpha = nv^\alpha = (n\gamma c, n\gamma\mathbf{v}) = (n'c, n'\mathbf{v}) \tag{1.45}$$

1.6.1 Conservation Laws

In order to conserve particle number, systems like in fluid mechanics obey the continuity equation

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0 \tag{1.46}$$

which holds in 3D, non-relativistic space. We can find a similar equation for N^α by taking the 4-gradient $\nabla = \frac{\partial}{\partial x^\alpha}$ so

$$\nabla N^\alpha = \frac{\partial N^\alpha}{\partial x^\alpha} = \frac{1}{c}\frac{\partial}{\partial t}(n\gamma c) + \frac{\partial}{\partial x^i}(\gamma n v_i) = \frac{\partial n'}{\partial t} + \nabla(n'\mathbf{v}) = 0 \tag{1.47}$$

Comma notation

Definition 1.6.2. Comma notation:

$$\nabla N^\alpha = \frac{\partial N^\alpha}{\partial x^\alpha} = \partial_\alpha N^\alpha = N^\alpha_{,\alpha} \tag{1.48}$$

Please stare at this and get the notation. A comma in the lower index position, followed by an index means differentiate the quantity N w.r.t the **contravariant** components x^α!

1.6.2 Energy-momentum tensor

$$T^{\mu\nu} = \rho v^\mu v^\nu, \tag{1.49}$$

where ρ is the mass density in IRF. We note $T^{\mu\nu}$ is a $\binom{2}{0}$-tensor.

- $T^{00} = \rho\gamma^2 c^2$ in IRF, and is equal to $\rho' c^2$ is the energy density in LT frame
- Other components T^{ij} for $i = x, y, z = 1, 2, 3$ is the flux of i-momentum in j-direction.
- In IRF, $\mathbf{v} = 0 \implies T^{00} = \rho' c^2 = \rho c^2$ is the *only* non-zero component of $T^{\mu\nu}$

1.6.3 Conservation Laws from $T^{\mu\nu}$

Consider the set of 4 equations

$$\partial_\mu T^{\mu\nu} = 0. \tag{1.50}$$

Let's look at $\nu = 0$ and $\nu = 1$. When $\nu = 0$, we have

$$\frac{1}{c}\frac{\partial}{\partial t}(\rho\gamma^2 c^2) + \frac{\partial}{\partial x^i}(\rho\gamma^2 c v_i) = 0 \iff \frac{\partial \rho'}{\partial t} + \nabla \cdot (\rho'\mathbf{v}) = 0 \tag{1.51}$$

We have recovered the continuity equation, which must be obeyed! Now, suppose $\nu = 1$. We must evaluate $\partial_\mu T^{\mu 1}$:

$$\frac{1}{c}\frac{\partial}{\partial t}(\rho\gamma c\gamma v_x) + \frac{\partial}{\partial x^i}(\rho\gamma^2 v_i v_x) = 0 \tag{1.52}$$

We expand the ∂_{x^i} term as follows:

$$\frac{\partial}{\partial x^i}(\rho\gamma^2 v_i v_x) = \rho'\left[v_x\frac{\partial v_i}{\partial x^i} + v_i\frac{\partial v_x}{\partial x^i}\right] + v_i v_x\frac{\partial \rho'}{\partial x^i} \tag{1.53}$$

Multiply the continuity equation by v_x, it now becomes

$$v_x \frac{\partial \rho'}{\partial t} + v_x \frac{\partial}{\partial x^i}(\rho \gamma^2 v_i) = v_x \frac{\partial \rho'}{\partial t} + v_x \left[v_i \frac{\partial \rho'}{\partial x^i} + \rho' \frac{\partial v_i}{\partial x^i} \right] = 0 \tag{1.54}$$

Since both Eq. (1.53) and Eq. (1.54) are equal to 0, we equate them to each other and notice terms actually start cancelling out once all the derivatives are expanded! This leaves us with

$$\rho' \frac{\partial v_x}{\partial t} + \rho' v_i \frac{\partial v_x}{\partial x^i} = 0 \tag{1.55}$$

for the $\nu = 1$ component. Similar holds for the other ν values. Thus $\nu = 1, 2, 3$ can be captured entirely in one derivative. This is nothing more than the **advective/material derivative** from fluid mechanics:

$$\rho' \frac{D\mathbf{v}}{Dt} = \rho' \left[\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right] \mathbf{v} = 0 \tag{1.56}$$

The fact this equals zero implies that there are no forces - sure. For an ideal fluid (no conduction, viscosity, radiation etc.) there will be a pressure flow ∇P, hence we really need

$$\rho' \frac{D\mathbf{v}}{Dt} = -\nabla P \tag{1.57}$$

So, it looks like our original $T^{\mu\nu}$ was slightly incorrect - it is now

Energy-Momentum Tensor

$$T^{\mu\nu} = \left(\rho + \frac{P}{c^2} \right) v^\mu v^\nu + P g^{\mu\nu} \tag{1.58}$$

The conservation laws in SR flat Minkowski space end up being as before. However in GR, we need to generalise to the covariant derivative. This is a tensor equation and holds in all reference frames - pretty cool!

1.7 Covariant derivative

We have been talking about many derivatives of coordinates and 1-forms so far. Earlier, we introduced the definition for a general $\binom{k}{l}$-tensor and its tensor product with another $\binom{p}{q}$-tensor. We now would like to combine these to talk about the **derivative** of vector fields along an arbitrary manifold. In effect, we will be generalising derivatives of functions in a Cartesian x, y plane that we are used to (and would love to have only have to do that...).

By Proposition 1.5.1, the components of the derivative is not the same as the derivative of the components. This generalises to tensors, where

$$\left(\frac{\partial \mathbf{v}}{\partial x^\alpha} \right)_\alpha \neq \frac{\partial v^\alpha}{\partial x^\alpha}$$

This is because the components vectors can change. Every vector $\mathbf{v} = v^\alpha \underline{e}_\alpha$. When differentiating, we must use the product rule:

$$\frac{\partial \mathbf{v}}{\partial x^\beta} = \frac{\partial}{\partial x^\beta} (v^\alpha \underline{e}_\alpha) = \frac{\partial v^\alpha}{\partial x^\beta} \underline{e}_\alpha + v^\alpha \frac{\partial \underline{e}_\alpha}{\partial x^\beta} \tag{1.59}$$

By homogeneity, the second term is a vector and still lies in the same tangent space of $\mathbf{v}$. Since the tangent space is locally Euclidean, we can describe it with a Euclidean vector space and thus we can express derivative in the second term above as

Christoffel Symbols

$$\frac{\partial \underline{e}_\alpha}{\partial x^\beta} = \Gamma^\gamma_{\ \alpha\beta}\underline{e}_\gamma \tag{1.60}$$

- The $\Gamma^\gamma_{\ \alpha\beta}$ are functions of x^α (**they also are locally-dependent**).
- They are called **connection coefficients/Christoffel symbols.**[1]

We can substitute this expression for the derivative into Eq. (1.59) and get

$$\frac{\partial \mathbf{v}}{\partial x^\beta} = \left(\frac{\partial v^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\ \gamma\beta}v^\gamma\right)\underline{e}_\alpha \tag{1.61}$$

What is important to note is that

$\partial_\beta v^\alpha$ and $\Gamma^\alpha_{\ \gamma\beta}$ are **not tensors**.

Remark. It helps to attempt to visualise what the Christoffel symbols represent geometrically (even if you couldn't care less). At a point x^α in a manifold M, its tangent space $T_{x^\alpha}M$ is spanned by some $\underline{e}_i$. If you move to some $x^\alpha + dx^\alpha$, then there is a new tangent space $T_{x^\alpha+dx^\alpha}M$ spanned by some basis $\underline{e}_i + d\underline{e}_i$. Since tangent spaces are locally Euclidean, we can define a dot product of vectors. Dot product $\underline{e}_\gamma$ on both sides of Eq. (1.60) to get

$$\Gamma^\gamma_{\ \alpha\beta} = \frac{\partial \underline{e}_\alpha}{\partial x^\beta} \cdot \underline{e}_\gamma$$

Hence, the *change* in the tangent space basis along each coordinate x^i is of the form $(\underline{e}_i + d\underline{e}_i - \underline{e}_i)/(x^\alpha + dx^\alpha - x^\alpha) \to \partial\underline{e}_i/\partial x^\alpha$. However this is just the actual change - we can then *project* this change along the different $\underline{e}_i$ and we see that the Christoffel symbols represent the derivatives of the basis along a basis direction.

As for the location of the indices, that doesn't have any geometric meaning (to my knowledge!), it comes out of having to satisfy index notation, and lowering and raising. Remember you can always lower and raise by the metric tensor.

Covariant derivative

Definition 1.7.1. The covariant derivative is a way to define a derivative along tangent vectors of a manifold. It is related to linear connections.

$$\frac{\partial \mathbf{v}}{\partial x^\beta} := v^\alpha_{;\beta}\underline{e}_\alpha \tag{1.62}$$

where

$$v^\alpha_{;\beta} = v^\alpha_{,\beta} + \Gamma^\alpha_{\ \gamma\beta}v^\gamma \tag{1.63}$$

[1]These come as a result of (linear) connections on pseudo-Riemannian manifolds. The maths is ridiculously abstract (Differential Geometry) but effectively allows us to "differentiate vector fields" on manifolds.

Proposition 1.7.1. *The Christoffel symbol is symmetric in the lower indices (covariant components).*

$$\Gamma^{\alpha}_{\gamma\beta} = \Gamma^{\alpha}_{\beta\gamma} \tag{1.64}$$

Proof. Prof. Tony Arber's notes attached as an Appendix TODO □

Theorem 1.7.1. *Levi-Civita Connection Formula*

$$\Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2} g^{\nu\gamma} \left(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu} \right) \tag{1.65}$$

Expanding out comma notation, this is (with dummy indices changed again)

$$\Gamma^{k}_{ij} = \frac{1}{2} g^{kl} \left(\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij} \right) \tag{1.66}$$

Remark. Equivalent equation for the 'Fundamental Lemma of Riemannian Geometry'. We are working on pseudo-Riemannian manifolds, but luckily there is a theorem that allows pseudo-Riemannian manifolds to have linear connections, every time, so we are fine.

Theorem 1.7.1 determines $\Gamma^{\gamma}_{\alpha\beta}$ in terms of only the metric $g_{\alpha\beta}$, removing the dependence to have to specify the Christoffel symbols in every tangent space.

1.7.1 1-forms

Now we want to be able to differentiate 1-forms. We have successfully differentiated vectors $v^{\alpha} \to v^{\alpha}_{;\beta}$, so for 1-forms we are trying to find an expression for $v_{\alpha;\beta}$.

We can consider some scalar invariant $\phi = p_{\alpha} v^{\alpha}$. Then $\phi_{;\beta} = \phi_{,\beta}$. The LHS evaluates to

$$\phi_{;\beta} = \frac{\partial p_{\alpha}}{\partial x^{\beta}} v^{\alpha} + p_{\alpha} \frac{\partial v^{\alpha}}{\partial x^{\beta}}$$

However, we can substitute in Eq. (1.61), do some re-indexing and

Covariant derivative of 1-forms

$$p_{\alpha;\beta} = p_{\alpha,\beta} - \Gamma^{\gamma}_{\alpha\beta} p_{\gamma} \tag{1.67}$$

1.8 Parallel Transport

Suppose you have a manifold M and within it, a path P parameterised by some variable λ so that $P \to x^{\alpha}(\lambda)$. At each point in that path, we have a vector $\mathbf{v}$ assigned (a vector field if you will), and we want to look at how $\mathbf{v}$ changes with respect to any path. Since each point p has its own local tangent space, we can use the regular product rule as before:

$$\frac{d\mathbf{v}}{d\lambda} = \frac{dv^{\alpha}}{d\lambda} \underline{e}_{\alpha} + v^{\alpha} \frac{d\underline{e}_{\alpha}}{d\lambda} \tag{1.68}$$

Now, we can use the chain rule since $\underline{e}_{\alpha}$ is a function of the coordinates x^{α}, which are functions of some parameter λ.

$$\frac{d\underline{e}_{\alpha}}{d\lambda} = \frac{dx^{\beta}}{d\lambda} \frac{d\underline{e}_{\alpha}}{dx^{\beta}}$$

However, we know what the Christoffel symbols are, Eq, (1.60), so substituting this in above:

$$\frac{d\mathbf{v}}{d\lambda} = \left(\frac{dv^\alpha}{d\lambda} + \Gamma^\alpha_{\gamma\beta}\frac{dx^\beta}{d\lambda}v^\gamma\right)\underline{e}_\alpha \tag{1.69}$$

The LHS is a scalar derivative of a vector: still a vector. Therefore the RHS must be a vector and we can write it as

$$\frac{d\mathbf{v}}{d\lambda} = \frac{Dv^\alpha}{D\lambda}\underline{e}_\alpha \tag{1.70}$$

This is the **intrinsic/total/absolute** derivative. But inside $Dv^\alpha/D\lambda$, we can expand $dv^\lambda/d\lambda$ with the chain rule

$$\begin{aligned}\frac{dv^\alpha}{d\lambda} &= \frac{dx^\beta}{d\lambda}\frac{dv^\alpha}{dx^\beta}\\ \frac{Dv^\alpha}{D\lambda} &= \left(\frac{\partial v^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\gamma\beta}v^\gamma\right)\frac{dx^\beta}{d\lambda} = v^\alpha_{;\beta}\frac{dx^\beta}{d\lambda}\end{aligned} \tag{1.71}$$

Now, we said λ is an arbitrary parameter, so we are free to choose it - we can set it equal to the proper time τ such that $dx^\beta/d\tau = u^\beta$, the 4-velocity. Therefore, the total derivative, measuring the change of a vector, equals to

$$\frac{Dv^\alpha}{D\tau} = v^\alpha_{;\beta}u^\beta \tag{1.72}$$

1.8.1 Parallel transport

Definition 1.8.1. A vector field along a curve is **parallel** if

$$\frac{d\mathbf{v}}{d\lambda} = \frac{Dv^\alpha}{D\lambda}\underline{e}_\alpha = 0, \tag{1.73}$$

i.e., a vector is being **parallel transported** if it satisfies this equation.
A vector field $V \in T_{x^\alpha(\lambda)}(M)$ on M is said to be **parallel** if it's parallel along every curve $x^\alpha(\lambda)$

This effectively generalises what it means for a vector to be parallel to its previous self as it moves along a path. Some questions may arise then:

1. Can you always have parallel vector fields as you move along a curve γ?
2. Can you parallel transport your own tangent vector (yes, see later).

The first question is answered by the Theorem of Parallel Translation, which says there exists a unique vector field for any curve, for every λ (assuming of course, the curve is parameterised by λ). Parallel transport is not always intuitive, and there are 3 main cases

- Flat space, (x, y, z) coordinates - v^α unchanged.
- Flat space, (r, θ) coordinates - $\mathbf{v}$ unchanged bu v^α changes.
- Curved space- v^α changes and $\mathbf{v}$ *may* change due to parallel transport.

To demonstrate the latter case, we look at a sphere, and 2 particular paths shown in Fig. 1.4. On the left, we see that a great circle route, with $\mathbf{v}$ perpendicular, keeps $\mathbf{v}$ unchanged. On the right however, $\mathbf{v}$ definitely changes, and this will be the case for any non-great circle path on a sphere. In both cases, v^α changes.
Remark. I glossed over this, but when we 'look at' a surface or manifold, we are *embedding* this into a higher spatial dimension. This makes life (and maths) easier for us. However, we live in the universe manifold, so we cannot look at our universe from a higher spatial dimension (anyone up to make the fourth spatial dimension?)

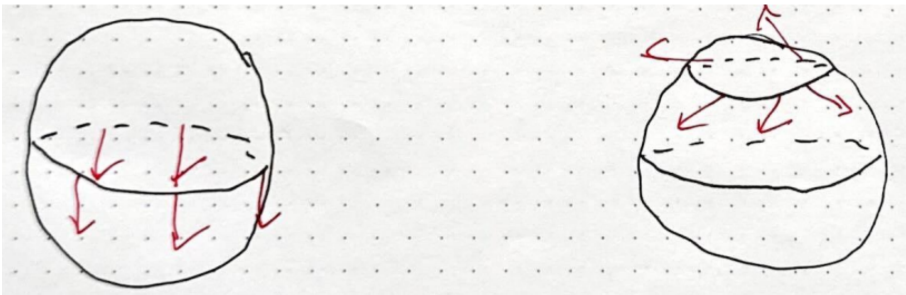


Figure 1.4: (left) Great circle path on a sphere with $\mathbf{v}$ pointing perpendicular to the path. (right) non-great circle path.

1.9 Geodesics

Definition 1.9.1. A line is **straight** if it parallel transports its own tangent vector

Examples:

- In flat, Euclidean space, every tangent vector is parallel to the line, so the tangent vector is indeed parallel transported
- Take any 2 points on a sphere - the 'straight line path' between those is any great circle path which the 2 points lie upon.

Since the parallel transport vector v^α equals the tangent vector u^α, a straight line satisfies $v^\alpha = u^\alpha$. Hence, instead of v^γ in Eq. (1.71), we substitute it for a generic $dx^\gamma/d\lambda$, and do the same for every occurrence of v^α. We therefore get

Geodesic equations

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\gamma\beta}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = 0 \tag{1.74}$$

Equivalently,

$$\ddot{x}^\alpha + \Gamma^\alpha_{\gamma\beta}\dot{x}^\beta\dot{x}^\gamma = 0 \tag{1.75}$$

Usually, $\lambda = \tau$ for massive particles - light is different.

Physically, the geodesic equations in Eq. (1.74) generalise the force-free motion $d\mathbf{v}/d\tau = 0$ in SR. In GR, gravity is not a force in the Newtonian sense, but it is a thing which distorts the space-time manifold, i.e. $g_{\alpha\beta}$ changes. This means

Solutions of the geodesic equation give orbits in GR.

To solve a problem, we first would need to obtain $g_{\alpha\beta}$. Then we can find $\Gamma^\gamma_{\alpha\beta}$ as per Theorem. 1.7.1 and then solve the geodesic equation.

- To find $g_{\alpha\beta}$, we need to specify 10 independent components. Remember in four dimensions, there are 16 total components, but it is symmetric and so we need only find the diagonal and one of the upper or lower triangles.
- The general, 4 dimensional $\Gamma^\gamma_{\alpha\beta}$ has 40 independent components.

1.9.1 Euler-Lagrange Equations

As you know, the Euler-Lagrange (EL) equations relate the position, momenta and energies of particles together. In general, the Lagrangian $L = L(t, x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n)$ meaning it has $2n+1$ independent variables for an n-body system. For this module, we will assume time-independence, so we have $2n$ components[2]. The positions must lie on the manifold M where the problem lies, and it also involves its derivatives, which lie in the tangent space, and so the Lagrangian must map from somewhere inside the tangent bundle TM and produce a real number (the energy difference). Hence, the Lagrangian, classically defined as $T - V$, generalises to

$$L : TM \to \mathbb{R} \qquad\qquad L(x^\alpha, \dot{x}^\alpha) = \frac{1}{2} h_{x^\alpha}(\dot{x}^\alpha) - V(x^\alpha), \tag{1.76}$$

where $V : M \to \mathbb{R}$ is the **potential function**, $(x^\alpha, \dot{x}^\alpha)$ are functions of λ and $h_{x^\alpha} : TM \to \mathbb{R}$ is a **kinetic energy function** with respect to the tangent space about x^α. We can of course still choose our own Lagrangian for a given problem, they don't have to be in the exact same form but must satisfy the same criteria. In our case, let's have

$$L(x^\alpha, \dot{x}^\alpha) = \frac{1}{2} g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta \tag{1.77}$$

Euler-Lagrange equation

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}_\alpha}\right) - \frac{\partial L}{\partial x_\alpha} = 0 \tag{1.78}$$

You can also write it with a superscript instead.

Substituting this L into the EL equation gives the geodesic equation.

Proof. I have definitely gotten indices mixed up somewhere so please tell me where they are wrong! We show this by direct substitution into the EL equation, and use superscripts instead of subscripts (multiplying by the metric tensor will correct everything).

$$\frac{\partial L}{\partial \dot{x}^\alpha} = g_{\alpha\mu} \dot{x}^\mu \tag{1.79}$$

We are effectively differentiating a square. Remember there is implicit summation, so you can always fix one of the indices as α and sum over the other. You do this twice, once for each index, but then you see by rearrangement of the double indices that you are double counting each term, hence the factor of 2, which cancels out the 1/2. Now, we differentiate this w.r.t λ and use the product and chain rules:

$$\frac{d}{d\lambda}(g_{\alpha\beta}\dot{x}^\beta) = \left(\frac{dg_{\alpha\beta}}{d\lambda}\dot{x}^\beta + g_{\alpha\beta}\frac{d^2 x^\beta}{d\lambda^2}\right) \tag{1.80}$$

$$= \frac{\partial g_{\alpha\beta}}{\partial x^\delta}\dot{x}^\delta \dot{x}^\beta + g_{\alpha\delta}\ddot{x}^\delta \tag{1.81}$$

$$= g_{\alpha\beta,\delta}\dot{x}^\delta \dot{x}^\beta + g_{\alpha\delta}\ddot{x}^\delta \tag{1.82}$$

where the chain rule was used in the second equality, as well as reduction down to funny notation. Replace $\alpha \to \mu, \beta \leftrightarrow \nu$ and we can split the $g_{\mu\nu,\delta}$ term into 2 terms and rearrange

[2] Time-independence is actually needed, since a time-dependent Lagrangian means a time-dependent Hamiltonian. By Noether's theorem, energy would NOT be conserved, which Einstein did not want.

indices: $g_{\mu\nu,\delta} = \frac{1}{2}\left(g_{\mu\nu,\delta} + g_{\nu\delta,\mu}\right)$ and we get

$$\frac{d}{d\lambda}(g_{\mu\beta}\dot{x}^{\beta}) = \frac{1}{2}\left(g_{\mu\nu,\delta} + g_{\nu\delta,\mu}\right)\dot{x}^{\delta}\dot{x}^{\nu} + g_{\mu\delta}\ddot{x}^{\delta}$$

We now must deal with the other derivative term - this is way easier, since the only term that is a function of x^{α} is the metric, and so

$$\frac{\partial L}{\partial x^{\delta}} = \frac{1}{2}\frac{\partial g_{\mu\beta}}{\partial x^{\delta}}\dot{x}^{\mu}\dot{x}^{\beta} = \frac{1}{2}g_{\mu\nu,\delta}\dot{x}^{\mu}\dot{x}^{\nu} \tag{1.83}$$

To correct the indices to be the same as Eq. (1.9.1), set $\nu \leftrightarrow \delta$ and put everything into the Euler-Lagrange Eq. (1.78):

$$\frac{1}{2}\left(g_{\mu\nu,\delta} + g_{\nu\delta,\mu}\right)\dot{x}^{\delta}\dot{x}^{\nu} + g_{\alpha\delta}\ddot{x}^{\delta} - \frac{1}{2}g_{\mu\nu,\delta}\dot{x}^{\mu}\dot{x}^{\nu} = 0 \tag{1.84}$$

$$\iff g_{\delta\mu}\ddot{x}^{\mu} + \underbrace{\frac{1}{2}\left(g_{\mu\nu,\delta} + g_{\nu\delta,\mu} - g_{\mu\delta,\nu}\right)}_{=\Gamma_{\mu\delta\nu}}\dot{x}^{\delta}\dot{x}^{\nu} = 0 \tag{1.85}$$

To complete the final step, we contract with the inverse metric tensor $g^{\mu\delta}$ on both sides to raise the μ index to get

$$\ddot{x}^{\mu} + \Gamma^{\mu}_{\delta\nu}\dot{x}^{\delta}\dot{x}^{\nu} = 0 \tag{1.86}$$

□

Solving the EL is usually easier than finding $\Gamma^{\delta}_{\alpha\beta}$ and solving the geodesic equation - mainly because you don't have to labour away to find 40 components and we all know how to solve an EL anyways.

Example: Schwarzchild metric for black holes

$$ds^2 = -c^2\left(1 - \tfrac{2GM}{c^2 r}\right)dt^2 + \left(1 - \tfrac{2GM}{c^2 r}\right)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \tag{1.87}$$

The Lagrangian L is then $L = g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}$:[3]

$$L = -c^2\left(1 - \frac{GM}{c^2 r}\right)\dot{t}^2 + \frac{\dot{r}^2}{\left(1 - \frac{2GM}{c^2 r}\right)} + r^2\dot{\theta}^2 + r^2\sin^2(\theta)\dot{\phi}^2 \tag{1.88}$$

From this point on, we can find the equations for each component, such as θ:

$$\frac{d}{d\lambda}\left(2r^2\dot{\theta}\right) - 2r^2\sin\theta\cos\theta\phi^2 = 0 \tag{1.89}$$

We can also find **constants of motion**. This occurs when the Lagrangian is explicitly **not** a function of one of the position variables. In the Schwarzchild case, this is ϕ, since no ϕ term occurs and so

$$r^2\sin^2(\theta)\dot{\phi}^2 = \text{constant} \tag{1.90}$$

[3]I ignored the factor of 1/2 for convenience and to match the lecture notes going forward.

1.9.2 Slow motion in a weak field

Slow motion implies $\gamma = 1$. Hence, $\dot{x}^{i=1,2,3}$ is negligible compared to $\dot{x}^0$ (remember $\dot{x}^0 \sim c$). Considering its geodesic equation,

$$\ddot{x}^\alpha + \Gamma^\alpha_{00}\dot{x}^0\dot{x}^0 = 0 \tag{1.91}$$

If the metric is stationary then all $g_{\alpha\beta,0} = 0$ in $\Gamma^\gamma_{\beta\alpha}$. Hence, $\Gamma^0_{00} = 0$. For the spatial coordinates, we have

$$\Gamma^i_{00} = \frac{1}{2}g^{\nu i}\left(g_{\nu 0,0} + g_{0\nu,0} - g_{00,\nu}\right)$$

For a weak field, we can treat the situation as a minor perturbation to Minkowski flat space time, so

$$g_{\alpha\beta} \sim \eta_{\alpha\beta} + h_{\alpha\beta}, \ |h_{\alpha\beta}| \ll 1 \tag{1.92}$$

$$\Gamma^i_{00} \sim \frac{1}{2}\eta^{\nu i}\left(-\frac{\partial g_{00}}{\partial x^\nu}\right) \tag{1.93}$$

Now, remember the Minkowski tensor is 1 in the spatial components, -1 otherwise. We can sum over ν, but note that $\eta^{\nu i} = 0$ whenever $\nu \neq i$, so we only keep $\nu = i$ terms. Hence, $-1 + 1 + 1 + 1 = 2$. $2 \times 1/2 = 1$. Therefore,

$$\Gamma^i_{00} = -\frac{\partial g_{00}}{\partial x^\nu} \tag{1.94}$$

Substitute this into the geodesic equation and we get

$$\frac{1}{c^2}\frac{d^2x^i}{dt^2} = \frac{1}{2}\frac{\partial}{\partial x^i}g_{00} \tag{1.95}$$

This looks like Newton's equation,

$$\frac{d^2x^i}{dt^2} = -\nabla\phi \tag{1.96}$$

, where ϕ is the gravitational potential (which remember, GR doesn't know about, everything is a result of manifold distortion). Equating coefficients,

$$g_{00} = -1 - \frac{2}{c^2}\phi \tag{1.97}$$

So Newton's theory of gravitation is a non-relativistic, low mass (field is treated as a perturbation) regime of GR.

1.10 Curvature

We have conquered tensors, covariant derivatives, parallel transport and the Christoffel symbols. We are going to start to put everything together as we discuss curvature.

1.10.1 Local flatness

Suppose we have a generic transformation

$$g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}g_{\mu\nu}$$

It is **always** possible to find $x^{\alpha'} = x^{\alpha'}(x^\alpha)$ such that $g_{\alpha'\beta'} = \eta_{\alpha'\beta'}$, but this is **locally around** x^α, since only the space near it can be considered 'locally flat'.

You can also choose $x^{\alpha'}$ such that $g_{\alpha\beta,\gamma} = 0$!

Definition 1.10.1. A frame with $g_{\alpha\beta} = \eta_{\alpha\beta}$ and $g_{\alpha\beta,\gamma} = 0$ are **local inertial frames**.

These are free falling by the equivalence principle. This implies space-time is locally flat.

What you cannot do, is find $x^{\alpha'}$ such that

$$\partial_\alpha \partial_\beta g_{\delta\gamma} = 0 \qquad\qquad g_{\delta\gamma,\alpha\beta} = 0$$

1.10.2 Riemann Curvature

Let us now consider a small region on a manifold M that is curved, that is parameterised by 2 coordinates x^1, x^2, such that the region ABCD is shown in Fig. 1.5. We consider our

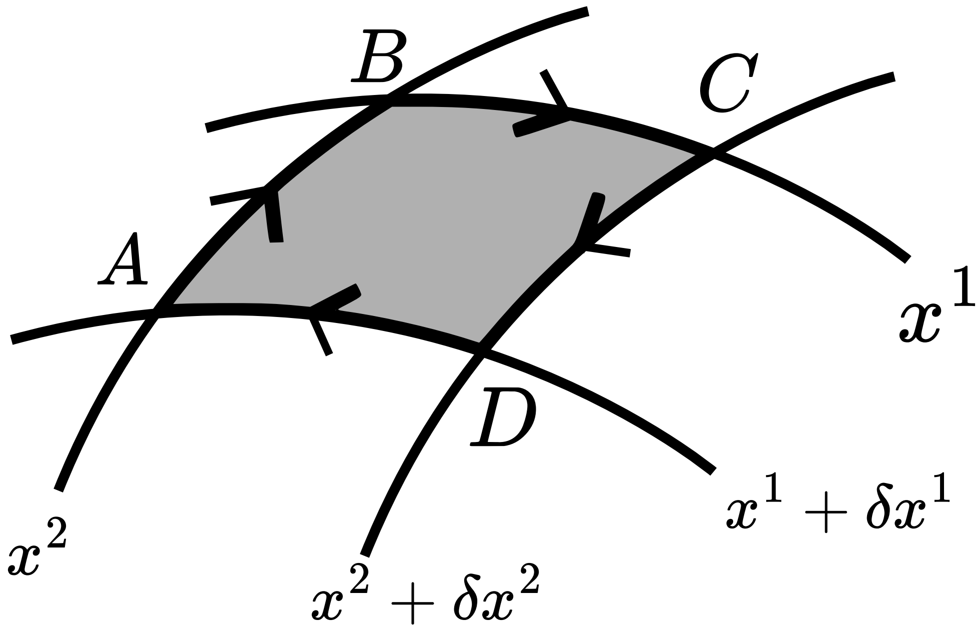


Figure 1.5: Region in curved space parameterised y contravariant coordinates (x^1, x^2).

parallel transport vector v^α, which remember, satisfies Definition 1.8.1, that is

$$\frac{Dv^\alpha}{Dt} = v^\alpha_{;\beta} = v^\alpha_{,\beta} + \Gamma^\alpha_{\beta\gamma} v^\beta = 0$$

So for our path $ABCD$ in Fig. 1.5, we evaluate each contribution separately and add them up:

$A \to B$**:**

$$\frac{\partial v^\alpha}{\partial x^1} = -\Gamma^\alpha_{\beta 1} v^\beta$$

We assume the variations δx^i are small and use a finite difference approximation

$$\frac{\partial v^\alpha}{\partial x^1} \approx \frac{v^\alpha(B) - v^\alpha(A)}{\delta x^1} \implies v^\alpha(B) = v^\alpha(A) - \left(\Gamma^\alpha_{\gamma 1} v^\gamma\right)_{x^2} \delta x^1 \tag{1.98}$$

Since path CD is the same as AB but in reverse, we get a similar expression with a sign difference and slightly different evaluation:

$$v^\alpha(D) = v^\alpha(C) + \left(\Gamma^\alpha_{\gamma 1} v^\gamma\right)_{x^2+\delta x^2} \delta x^1 \tag{1.99}$$

Adding up the 2 contributions: $AB + CD$ gives

$$
\begin{aligned}
AB + CD &= \left(v^\alpha(B) - v^\alpha(A)\right) + \left(v^\alpha(D) - v^\alpha(C)\right) && (1.100)\\
&= \left(\Gamma^\alpha_{\gamma 1} v^\gamma\right)_{x^2+\delta x^2} \delta x^1 - \left(\Gamma^\alpha_{\gamma 1} v^\gamma\right)_{x^2} \delta x^1 && (1.101)\\
&= \left[\frac{\partial}{\partial x^2}\left(\Gamma^\alpha_{\beta 1} v^\beta\right)\right] \delta x^1 \delta x^2. && (1.102)
\end{aligned}
$$

We can repeat the same calculation for the paths BC and DA, with the same logic. Only this time, the derivative is over x^2, so the final term will be a partial derivative w.r.t x^1. Additionally, the Christoffel symbol will be $\Gamma^\alpha_{\beta 2}$ instead. You will also introduce a minus sign in front because C and A are now at the ends of the 2 paths, $B\underline{C}$ and $D\underline{A}$. Thus, summing over $ABCD$:

$$
\delta v^\alpha = \delta x^1 \delta x^2 \left[-\frac{\partial}{\partial x^1}\left(\Gamma^\alpha_{\beta 2} v^\beta\right) + \frac{\partial}{\partial x^2}\left(\Gamma^\alpha_{\beta 1} v^\beta\right)\right] \tag{1.103}
$$

Now, let's expand the derivatives a wee bit with the product rule:

$$
\begin{aligned}
\frac{\partial}{\partial x^1}\Gamma^\alpha_{\beta 2} v^\beta &= \left(\partial_{x^1}\Gamma^\alpha_{\beta 2}\right) v^\beta + \Gamma^\alpha_{\beta 2}\left(\partial_{x^1} v^\beta\right) && (1.104)\\
\frac{\partial}{\partial x^2}\Gamma^\alpha_{\beta 1} v^\beta &= \left(\partial_{x^2}\Gamma^\alpha_{\beta 1}\right) v^\beta + \Gamma^\alpha_{\beta 1}\left(\partial_{x^2} v^\beta\right) && (1.105)
\end{aligned}
$$

Collecting everything into δv^α and rearranging in a suggestive form:

$$
\delta v^\alpha = \delta x^1 \delta x^2 \left[\left(\partial_{x^2}\Gamma^\alpha_{\beta 2} - \partial_{x^1}\Gamma^\alpha_{\beta 2}\right) + \left(\Gamma^\alpha_{\beta 1}\partial_{x^2} - \Gamma^\alpha_{\beta 2}\partial_{x^1}\right)\right] v^\beta
$$

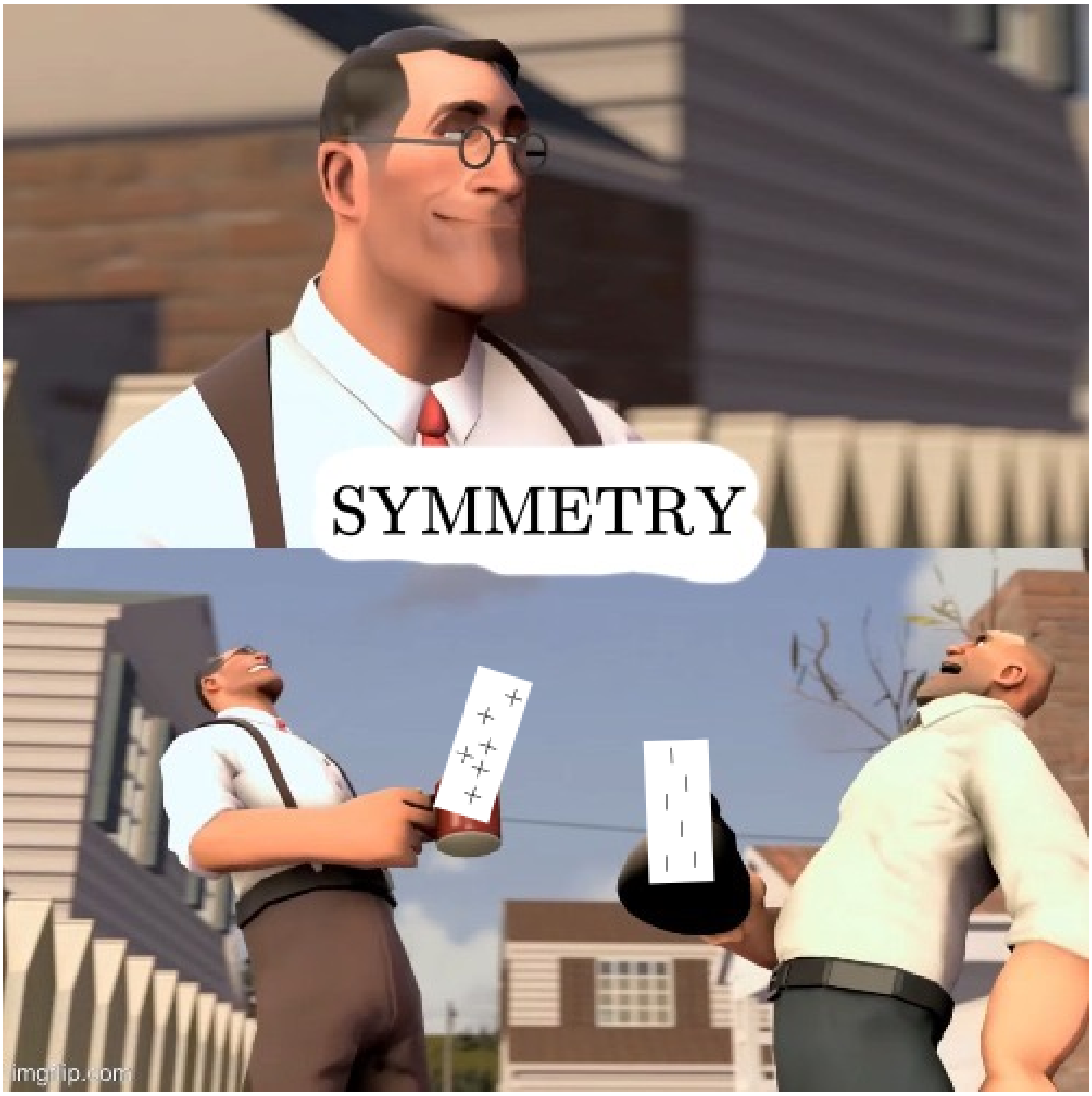
But we can have a further simplification, particularly of the $\Gamma^\alpha_{\beta\gamma}\partial_{x^\mu} := \Gamma^\alpha_{\beta\gamma}\partial_\mu$ terms. Recall that we are parallel-transporting v^α, so it satisfies

$$
\left(\partial_{x^\alpha} v^\beta + \Gamma^\alpha_{\beta\gamma} v^\gamma\right)\frac{dx^\beta}{d\lambda} = 0
$$

Now, since we are moving along the path, $dx^\beta/d\lambda$ is certainly not zero, so by equivalence, $\partial_\alpha v^\beta = -\Gamma^\alpha_{\beta\gamma} v^\gamma$. Substituting this in to Eq. (1.10.2) and ensuring the indices are arranged appropriately, we have

$$
\delta v^\alpha = \delta x^1 \delta x^2 \left[\left(\partial_{x^2}\Gamma^\alpha_{\beta 2} - \partial_{x^1}\Gamma^\alpha_{\beta 2}\right) - \left(\Gamma^\alpha_{1\sigma}\Gamma^\sigma_{\beta 2} + \Gamma^\alpha_{2\sigma}\Gamma^\sigma_{\beta 1}\right)\right] v^\beta
$$

Now, let's count indices here. The $\partial_{x^\mu}\Gamma^\alpha_{\beta\gamma} = \partial_\mu\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma,\mu}$ terms have 4 independent indices, and the derivative here is the contravariant 4-gradient of a Christoffel symbol which *is* a tensor. Pretty cool. Similarly, the $\Gamma^\alpha_{1\sigma}\Gamma^\sigma_{\beta 2}$ is also a tensor by Lemma 1.3.1. So what we really have constructed are components of a rank-$\binom{1}{3}$ tensor. However, we can go further. We only considered a 2-component manifold - if we have an arbitrary manifold with coordinates $(x^1, \ldots, x^n)$ and an arbitrary path in that manifold, using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we have equal amounts of pluses and minuses. We can also generalise our indices to μ's and ν's (from 1's and 2's) and we get a new tensor:

Riemann curvature tensor

$$R^{\alpha}_{\beta\mu\nu} = \Gamma^{\alpha}_{\beta\nu,\mu} - \Gamma^{\alpha}_{\beta\mu,\nu} + \Gamma^{\alpha}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\alpha}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu} \tag{1.106}$$

1.10.3 Properties of the Riemann curvature tensor

In this subsection, we will state some properties of $R^{\alpha}_{\beta\mu\nu}$ without proof, that will be useful later

Lemma 1.10.1. *Let $R_{\alpha\beta\gamma\delta} = g_{\alpha\mu}R^{\mu}_{\beta\gamma\delta}$ be the fully covariant Riemann tensor. Then*

- *It is antisymmetric when swapping individual pairs of lower indices: $R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\delta\gamma}$*
- *It is symmetric when swapping the two pairs of indices: $R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta} =$*
- *It satisfies the First Bianchi Identity: $R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta} = 0$*
- *It satisfies the Second/Differential Bianchi Identity $R_{\alpha\beta\gamma\delta;\mu} + R_{\alpha\beta\mu\gamma;\delta} + R_{\alpha\beta\delta\mu;\gamma} = 0$. It is important in calculating the divergence of the Ricci tensor and thus is needed for Einstein's field equations*

Remark. In lectures and in the handout, the two Bianchi identities are referred to as the same thing. Just for clarity, I have distinguished them here with the same convention as MA4C0 Differential Geometry.

The Riemann curvature tensor may differ by a $\pm$ in different sources.

1.10.4 Ricci tensor

GR doesn't need the full Riemann curvature tensor, but we can consider its traces

Ricci curvature tensor

The Ricci curvature tensor $R_{\beta\nu} \in \mathcal{T}^0_2(M)$ on a manifold M is defined as

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\alpha\mu,\nu} + \Gamma^{\alpha}_{\alpha\sigma}\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\alpha}_{\nu\sigma}\Gamma^{\sigma}_{\mu\alpha}, \tag{1.107}$$

i.e., it's the trace in the second index. To get this, it will help to start from $R^{\alpha}_{\beta\alpha\nu}$ and evaluate it. Then relabel $\beta \to \mu$ and use symmetry in lower indices of Christoffel symbols to get this.

Lemma 1.10.2. *$R_{\mu\nu} = R_{\nu\mu}$, i.e. the Ricci tensor is symmetric.*

Proof. Follows directly from the definition and properties of Riemann curvature. □

The Ricci tensor is unique up to $\pm$ sign due to the same reason for the Riemann tensor, leading to the following

Lemma 1.10.3. *Further contractions of the Riemann tensor:*

- $R^{\beta}_{\beta\mu\nu} = 0$
- $R^{\alpha}_{\beta\nu\alpha} = -R^{\alpha}_{\beta\alpha\nu}$

Proof. The first point follows from the Riemann tensor and the second point from Lemma 1.10.1. □

Earlier, we found an expression for the Christoffel symbols entirely in terms of the metric $g_{\alpha\beta}$ and its derivatives. Consequently, the Riemann curvature and Ricci curvature are also entirely determined by the metric and its derivatives.

In the LIF (local inertial frame), $g_{\alpha\beta} = \eta_{\alpha\beta}$ and $g_{\alpha\beta;\gamma} = 0$. $R_{\mu\nu}$ then becomes much simpler, as the product of Christoffel symbols becomes 0

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu;\alpha} - \Gamma^{\alpha}_{\alpha\nu,\mu} \tag{1.108}$$

Proposition 1.10.1. *Contracted Bianchi Identity. Let $R_{\beta\gamma}$ be the Ricci curvature tensor. It satisfies the following equation:*

$$R_{\beta\gamma;\mu} + R^{\delta}_{\beta\mu\gamma;\delta} - R_{\beta\mu;\gamma} = 0. \tag{1.109}$$

Note the middle term is a covariant derivative of the Riemann tensor. Funnily enough, this is actually its divergence although it doesn't look like a normal divergence from vector calculus. That being said, I think we are far enough away from vector calculus at this point so nothing is 'normal', it's just 'fine'.

Corollary 1.10.1. *Let $R^{\alpha\beta}$ be the contravariant Ricci curvature tensor. It satisfies*

$$R^{\alpha\beta}_{;\alpha} = \frac{1}{2} R_{;\mu} g^{\mu\beta} \tag{1.110}$$

Proof. Assuming the contracted Bianchi identity, contract β and γ withj $g^{\beta\gamma}$:

$$\begin{aligned} g^{\beta\gamma}\left(R_{\beta\gamma;\mu} + R^{\delta}_{\beta\mu\gamma;\delta} - R_{\beta\mu;\gamma}\right) &= 0 \\ R_{;\mu} - R^{\delta}_{\mu;\delta} - R^{\gamma}_{\mu;\gamma} &= 0. \end{aligned}$$

Changing $\delta \to \alpha$ and multiplying by $g^{\mu\beta}$ then shows that

$$R^{\alpha\beta}_{;\alpha} = \frac{1}{2} R_{;\mu} g^{\mu\beta}$$

as required. □

Definition 1.10.2. The **Ricci scalar** is a full contraction of the Ricci tensor

$$R = g^{\alpha\beta} R_{\alpha\beta} \tag{1.111}$$

Because it is a scalar, it is invariant under all frames. Therefore, a neat trick is to always find the easiest frame to calculate this in.

The Ricci and Riemann tensors can also tell us about whether the space is curved.

- $R_{\mu\nu\alpha\beta} = 0 \implies$ space is not curved and flat everywhere
- $R_{\mu\nu} = 0, R_{\mu\nu\alpha\beta} \neq 0 \implies$ space is **not necessarily flat** - space can still curve in the intuitive sense - **Ricci-flat**
- Both tensors non-zero, definitely not flat in any way

1.11 Einstein Field Equations

Now we can use the previous section to transform Newton's equation Eq. (1.1) from the beginning of the module, into something that is GR-ready. This new theory must satisfy

- Relativity (including in the SR sense)
- Tensorial (holds in all frames and coordinate transforms)
- Accounts for curvature of space-time
- Obeys all the conservation laws derived from the energy-momentum tensor
- Satisfies geodesic equation for free motion
- Satisfies the slow motion limit

Newton's equations tells us the Laplacian of some potential $\nabla^2\phi$ is responsible for the gravitational force. The Laplacian measures curvature, and so we expect this to be replaced with the Ricci tensor. We also know that $\nabla^2\phi$ is a slow motion limit of GR as in Section 1.9.2 (effectively, a boundary condition). Now, the right-hand side of Newton's equation, $4\pi G\rho$ is a function of mass-density. This will be replaced with energy density, i.e. a function of $T^{\mu\nu}$. In true physicist fashion, we *try* some possibilities and then find the first one which works.

1.11.1 Finding Einstein's equations

Try

$$R^{\alpha\beta} = kT^{\alpha\beta}, \tag{1.112}$$

where $k \in \mathbb{R}$ is a constant. The tensorial ranks on both sides are the same - which is good. We have satisfied general curvature with the Ricci tensor and ensured the energy density ρc^2 is involved. All is good, except one big elephant.

Problem Whilst $T^{\alpha\beta}_{;\beta} = 0$, we know by Corollary 1.10.1 that

$$R^{\alpha\beta}{}_{;\alpha} = \frac{1}{2}R_{;\alpha}g^{\alpha\beta} = \frac{1}{2}\left(\partial_\alpha R\right)g^{\alpha\beta} \neq 0$$

So the RHS of Eq. (1.112) is 0 but the LHS is non-zero - that is bad.

Try Our problem was that we had an extra term that was non-zero when we covariantly-differentiated both sides. So let's just make a new term, which when covariantly differentiated, cancels out:

$$G^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta} = kT^{\alpha\beta} \tag{1.113}$$

Indeed, if we apply the covariant derivative to the LHS:

$$G^{\alpha\beta}_{;\alpha} = R^{\alpha\beta}_{;\alpha} - \frac{1}{2}R_{;\alpha}g^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta}_{;\alpha} \tag{1.114}$$

by the product rule. Indeed, the first two terms cancel since $R_{;\alpha} = R_{,\alpha}$ since R is a scalar. The third term is always 0 since $g^{\alpha\beta}_{;\alpha} = 0$. It also still satisfies some of our requirements, so this is good - there is one issue though, what is k?

1.11.2 Finding the constant

We have definitely satisfied conservation laws, SR, curvature, geodesics (because of the Riemann tensor) and of course, it is a tensorial equation. We require an actual value of k. The one thing we need to check, is consistency with the slow-motion limit, i.e. Newtonian gravity.

We contract Eq. (1.113) with $g_{\alpha\beta}$:

$$g_{\alpha\beta}R^{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}g^{\alpha\beta} = kg_{\alpha\beta}T^{\alpha\beta}. \tag{1.115}$$

On the RHS, define $T = g_{\alpha\beta}T^{\alpha\beta}$ - this is a scalar. On the LHS, we can use Proposition 1.3.2 to get that this is $\delta^\alpha_\alpha = 4$. Hence

$$R - 2R = kT \implies R = -kT$$

Using Eq. (1.113), we can rewrite everything in terms of $R^{\alpha\beta}$

$$R^{\alpha\beta} + \frac{1}{2}kTg^{\alpha\beta} = kT^{\alpha\beta} \implies R^{\alpha\beta} = k\left(T^{\alpha\beta} - \frac{1}{2}Tg^{\alpha\beta}\right).$$

In the Newtonian limit, $P/c^2 \ll \rho$ so $T_{\alpha\beta} \sim \rho u_\alpha u_\beta$. Now to find some terms:

$$T = g^{\alpha\beta}T_{\alpha\beta} = \rho g^{\alpha\beta}u_\alpha u_\beta = -\rho c^2 \implies \boxed{T = -\rho c^2}.$$

In the weak field limit, $g_{\alpha\beta} \sim \eta_{\alpha\beta} \implies \boxed{g_{00} = -1}$. Since things move slowly, $c \gg v_{i=x,y,z} \implies u^0 \gg u^{i=1,2,3} \implies \boxed{T^{00} = \rho c^2}$.

Evaluating R_{00}:

$$R_{00} = \Gamma^\alpha_{00,\alpha} - \Gamma^\alpha_{\alpha 0,0} + \Gamma^\alpha_{\alpha\sigma}\Gamma^\sigma_{00} - \Gamma^\alpha_{0\sigma}\Gamma^\sigma_{0\alpha}$$

We approximate the metric in the weak field limit as $g_{\alpha\beta} \approx \eta_{\alpha\beta} + h_{\alpha\beta}$ where $|h_{\alpha\beta}| \ll 1$. Additionally, Newtonian gravity is **time-independent** so $\Gamma^\alpha_{\alpha 0,0} = 0$. In Section 1.9.2, we found $\Gamma^i_{00} = c^{-2}\phi_{,i}$:

$$\begin{aligned} R_{00} &= \frac{1}{c^2}\phi_{,ii} = \frac{1}{c^2}\nabla^2\phi \\ R_{00} &= k\left(T_{00} - \frac{1}{2}Tg_{00}\right) = \frac{k}{2}\rho c^2 = -\frac{4\pi G}{c^2}\rho \\ \implies & \boxed{k = \frac{8\pi G}{c^4}} \end{aligned} \tag{1.116}$$

so the full Einstein Field Equations are

Einstein Field Equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{1.117}$$

- 10 coupled, non-linear PDEs - very few analytical solutions and best left for simulations
- Involves $\Gamma^\alpha_{\beta\gamma}$ which are functions of $g_{\alpha\beta}$ - 40 components
- Conserves particle number, energy and mass if $g_{\alpha\beta}$ is time-independent
- Still a postulate and must be constantly tested against observations

Chapter 2

Schwarzchild Metric

Actually solving the full set of field equations analytically is pretty hard. It is easier to solve symmetric problems with good assumptions to make life easier, to make finding ds^2 and thus $g_{\alpha\beta}$ easier.

2.1 Deriving the Schwarzchild metric

We assume spherically symmetric and pure vacuum conditions. The line element on a sphere is

$$ds^2 = r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) = r^2 d\Omega^2$$

where Ω is the unit of solid angle. Now let (r, t) vary, constrained such that spherical symmetry still holds

$$ds^2 = \alpha(r,t)dt^2 + \beta(r,t)dtdr + \gamma(r,t)dr^2 + \delta(r,t)d\Omega^2 \tag{2.1}$$

We look for solutions which are

time-reversible AND time-independent

Metrics with both these properties are called **static**. In particular
Theorem 2.1.1. ***Birkhoff's theorem***. *Any spherically symmetric solution of Einstein's field equations static and asymptotically flat, i.e. far away from a massive body, space-time looks Minkowski.*

A particular corollary of this theorem is
Corollary 2.1.1. *The most general metric satisfying Birkhoff's theorem is*

$$ds^2 = -\alpha(r)dt^2 + \gamma(r)dr^2 + \delta(r)d\Omega^2 \tag{2.2}$$

Additionally, our metric must also satisfy Newton's equations. In lectures, the derivation is glossed over, but I have put it here for completeness. First, we know $\delta(r) = r^2$ because it must be spherically symmetric. For fixed dt, dr we have $ds = rd\Omega = rd\phi$ around the equator, $\theta = \pi/2$. For the other components, the calculation involves computing many Christoffel symbols and components of the Ricci tensor - most of which are 0. So, the non-zero components are presented here.

$$
\begin{aligned}
\Gamma^0_{0r} = \Gamma^0_{r0} &= \frac{1}{2} g^{0\beta} \left(\partial_0 g_{\beta r} + \partial_r g_{0\beta} - \partial_\beta g_{0r} \right) \\
&= \frac{1}{2} g^{00} \left(\partial_r g_{00} \right), \\
&= \frac{1}{2\alpha} \partial_r \alpha.
\end{aligned} \tag{2.3}
$$

Here are all the non-zero Christoffel symbols (too lazy to type the working out zzz)

$$
\begin{aligned}
&\Gamma^0_{0r} = \Gamma^0_{r0} = \frac{\alpha'}{2\alpha}, \quad \Gamma^r_{00} = \frac{\alpha'}{2\gamma}, \quad \Gamma^r_{rr} = \frac{\gamma'}{2\gamma}, \quad \Gamma^r_{\theta\theta} = \frac{-r}{\gamma}, \quad \Gamma^r_{\phi\phi} = \frac{-r\sin^2\theta}{\gamma}, \\
&\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}, \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \quad \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r}, \quad \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \frac{\cos\theta}{\sin\theta}.
\end{aligned} \tag{2.4}
$$

Using these, we can find the non-zero components of the Ricci tensor Eq. (1.107):

$$
\begin{aligned}
R_{00} &= \frac{1}{2\gamma} \left[\alpha'' - \frac{1}{2\alpha} \left(\alpha'\right)^2 - \frac{1}{2\gamma} \alpha'\gamma' + \frac{2}{r}\alpha' \right], \\
R_{rr} &= \frac{-1}{2\alpha} \left[\alpha'' - \frac{1}{2\alpha} \left(\alpha'\right)^2 - \frac{1}{2\gamma} \alpha'\gamma' - \frac{2\alpha}{r\gamma}\gamma' \right], \\
R_{\theta\theta} &= 1 - \frac{1}{\gamma} - \frac{r}{2\alpha\gamma}\alpha' + \frac{r}{2\gamma^2}\gamma', \\
R_{\phi\phi} &= R_{\theta\theta}\sin^2\theta.
\end{aligned} \tag{2.5}
$$

Next, we need to find the Ricci scalar Eq. (1.10.2)

$$
\begin{aligned}
R = g^{\mu\nu} R_{\mu\nu} &= g^{00} R_{00} + g^{rr} R_{rr} + g^{\theta\theta} R_{\theta\theta} + g^{\phi\phi} R_{\phi\phi}, \\
&= \frac{1}{\alpha\gamma} \left[-\alpha'' + \frac{1}{2\alpha} \left(\alpha'\right)^2 + \frac{1}{2\gamma}\alpha'\gamma' - \frac{2}{r}\alpha' + \frac{2\alpha}{r\gamma}\gamma' \right] + \frac{2}{r^2} \left(1 - \frac{1}{\gamma} \right).
\end{aligned} \tag{2.6}
$$

It now time to plug everything into the field equations Eq. (1.117). Since we are looking in the vacuum region where $T_{\mu\nu} = 0$. This is convenient - it stops us having to calculate another tensor, and tensorial equations hold in all reference frames and in every coordinate inside a frame. Therefore, the 00-component of Eq. (1.117) is

$$
R_{00} - \frac{1}{2} R g_{00} = \frac{\alpha}{r^2} \partial_r \left[r \left(1 - \frac{1}{\gamma} \right) \right] = 0, \quad \Rightarrow \quad r \left(1 - \frac{1}{\gamma} \right) = \text{ const. } \equiv 2m \tag{2.7}
$$

The rr-equation reads

$$
R_{rr} - \frac{1}{2} R g_{rr} = \frac{1}{r\alpha}\alpha' - \frac{\gamma}{r^2} \left(1 - \frac{1}{\gamma} \right) = 0
$$

and with the help of the solution for γ reduces to

$$
\frac{\alpha'}{\alpha} = \frac{\frac{2m}{r^2}}{1 - \frac{2m}{r}}
$$

In particular, the first solution which comes out of both equations is

$$
\alpha = 1 - \frac{2m}{r} \tag{2.8}
$$

Remark. You may want to check this is consistent with the $\theta\theta, \phi\phi$ equations - but to be honest, checking everything and doing the algebra is just long, not worth it. I got to about 3 pages on my tablet for the $\theta\theta$ equation before calling it quits because it's so long and you get nothing out of it.

Now, the final thing we need to check is consistency in the weak field limit, see Section 1.9.2. In particular, Eq. (1.97). Hence,

$$g_{00} = \left(1 - \frac{2m}{r}\right) = -1 - \frac{2}{c^2}\phi \tag{2.9}$$

We see that $m = GM/c^2$ for consistency with the Newtonian potential ϕ. We therefore arrive at the Schwarzchild metric

Schwarzchild metric

$$ds^2 = -c^2\left(1 - \frac{2GM}{rc^2}\right)dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1}dr^2 + r^2 d\Omega^2 \tag{2.10}$$

where M is the mass of the central body.

2.2 Schwarzchild Radius and Event Horizon

Definition 2.2.1. The **Schwarzchild radius** R_s is

$$R_s = \frac{2GM}{c^2} \tag{2.11}$$

g_{rr} *diverges* at this value.

Lemma 2.2.1. *This type of singularity is a* ***coordinate singularity****, namely we should be able to find some transformation (potentially non-linear) where g_{rr} no longer diverges at R_s*

Proof. Proof not covered explicitly in lectures or the problem sheet (so is non-examinable), but the new set of coordinates are called **Eddington-Finkelstein** coordinates. You replace the Schwarzchild time component with a coordinate that parameterises the radial null geodesics $t_\pm(r)$:

$$t_\pm(r) = \pm(r + 2m\log|r - 2m| + C)$$

where C is a constant that encodes the initial position r_0, so that $t_\pm(r_0) = 0$ and $t_\pm(r)$ represents the time taken for the light to reach a radial coordinate r from the frame of reference of an observer at infinity. The $+$ indicates an outgoing particle, whilst $-$ for ingoing. □

However, there is another singularity in Eq. (2.10), namely $r = 0$. This is the **black hole**. Inside $r < R_s$, $ds^2 = g_{tt}dt^2 + g_{rr}dr^2$ and the angular components are 0 (a person entering the black hole, and light, both move along radial geodesics). Weirdly though, the time and radial components **change sign**

$$g_{tt} > 0 \qquad\qquad g_{rr} < 0 \tag{2.12}$$

but for time-like particles (massive), we have $ds^2 < 0$. Light, of course, remains travelling along null geodesics. But then the particle is forced forever to have $dr < 0$, so it is moving **forever inwards to** $r = 0$.

2.3 Schwarzchild Orbits

From now on, we define

$$\mu := \frac{GM}{c^2} = \frac{1}{2}R_s$$

The Lagrangian $L = g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta$ becomes

$$L = -c^2\left(1 - \frac{2\mu}{r}\right)\dot{t}^2 + \left(1 - \frac{2\mu}{r}\right)\dot{r}^2 + r^2\left(\dot{\theta}^2 + \sin^2(\theta)\dot{\phi}^2\right) \tag{2.13}$$

Now, geodesics are found from solving the EL equation along a path parameterised by λ, and if L is not *explicitly* a function of some variables, then $\partial L/\partial\dot{x}^\alpha$ is a constant of motion. We see above that ϕ, t do not explicitly appear and so

$$r^2\sin^2\theta\dot{\phi}^2 := h \qquad\qquad \left(1 - \frac{2\mu}{r}\right)\dot{t} = k \tag{2.14}$$

so h, k are constants and h is the GR analogue of angular momentum.
Remark. Since h has dimensions of length. It is associated to the symmetry under $\phi \to \phi+$ const., which in classical mechanics yields conservation of the z-component of angular momentum.

For k, note in the limit $r \to \infty, k = \dot{t}$ and $g_{\alpha\beta} = \eta_{\alpha\beta}$, so $k = dt/d\tau = \gamma$ ($\lambda = \tau$, the proper time) and total energy is $E = \gamma mc^2 = kmc^2$. In general, $E = kmc^2$ and $k < 0$ is allowed because of gravity. In Newtonian gravity, potential energy is considered negative (because it places the object experiencing gravity in a sort of 'bound state').

We now want to find equations for $\dot{\theta}, \dot{r}$. The question is *how* to do it in a way that avoids long calculations. Well, if we step back for a moment and consider the θ equation. Then $L = r^2\left(\dot{\theta}^2 + \sin^2(\theta)\dot{\phi}^2\right)$. Suppose we have a perturbation of the form $\epsilon^\mu = (0, 0, \epsilon(\tau), 0)$. We can replace $\theta \to \theta + \epsilon$ in L and get

$$L = r^2\left[\left(\frac{d(\theta+\epsilon)}{d\tau}\right)^2 + \dot{\phi}^2\sin^2(\theta+\epsilon)\right]$$

We can expand the terms, using a Taylor series for the $\sin^2(\theta+\epsilon)$ and ignoring terms above first order:

$$L = r^2\left[\dot{\theta}^2 + \dot{\epsilon}^2 + 2\dot{\epsilon}\dot{\theta} + \dot{\phi}^2\left(\sin\theta + \epsilon\cos\theta\right)^2\right] + \dots$$

The metric along such an ϵ^μ (ignoring squares of derivatives) is

$$ds^2 = \left[-c^2\left(1 - \frac{2\mu}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \frac{1}{1 - \frac{2\mu}{r}}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(2\frac{d\theta}{d\tau}\frac{d\epsilon}{d\tau} + 2\epsilon\sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2\right) + \cdots\right]d\tau^2 \tag{2.15}$$

$$= \left[-1 + r^2\left(2\frac{d\theta}{d\tau}\frac{d\epsilon}{d\tau} + 2\epsilon\sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2\right) + \cdots\right]d\tau^2 \tag{2.16}$$

and hence the 'distance' (c times proper time) along the nearby curve is

$$\int ds^2 = \int d\tau + \int \epsilon\left[\frac{d}{d\tau}\left(r^2\frac{d\theta}{d\tau}\right) - \sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2\right]d\tau + \cdots$$

Setting the first order difference to zero we obtain

$$\frac{d}{d\tau}\left(r^2\frac{d\theta}{d\tau}\right)-\sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2=0$$

which is automatically satisfied if $\boxed{\theta=\pi/2}$. Indeed the cosine term will be 0, and as θ is fixed, all derivatives are 0. By spherical symmetry, this particular solution is enough to understand all of the geodesics. Another way to think of this is conservation of orbital angular momentum, where we align the spherical coordinates in the direction of the orbital angular momentum vector. Since this vector will be perpendicular to the direction of motion (think of $\mathbf{L}=\mathbf{r}\times\mathbf{p}$ in Newtonian physics), we can consider motion in the plane normal to this vector, which is the equatorial plane.

Now, we have to find the equation for $\dot{r}$. We could do the same thing, but luckily we have the fact that $g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta=-c^2$ for a massive particle. Combining everything so far, we have a set of 3 equations

$$\begin{aligned}&\left(1-\frac{2\mu}{r^2}\right)t=k\\&r^2\dot{\phi}=h\\&c^2\left(1-\frac{2\mu}{r}\right)t^2-\left(1-\frac{2\mu}{r}\right)^{-1}\dot{r}^2-r^2\dot{\phi}^2=c^2\end{aligned}\tag{2.17}$$

which when combined produces

$$\dot{r}^2+\frac{h^2}{r^2}\left(1-\frac{2\mu}{r}\right)-2\mu\frac{c^2}{r}=c^2\left(k^2-1\right)\tag{2.18}$$

This is the GR version of kinetic + potential = total energy, for motion along radial coordinate r. Additionally, compared to Newtonian potential energy, we have a term $\propto r^{-3}$. So we have walked through the trenches for this. Hopefully, it means something.

If we divide Eq. (2.18) by 2 and reinsert μ in Eq. (2.3), we can see that there is an **effective potential**:

$$V(r)=\frac{h^2}{2r^2}\left(1-\frac{2GM}{rc^2}\right)-\frac{GM}{r},\tag{2.19}$$

so we can rewrite Eq. (2.18) as

$$\dot{r}^2+2V(r)=c^2(k^2-1).\tag{2.20}$$

2.3.1 Types of orbits

Newton orbits

The effective potential reduces to

$$V(r)=\frac{h^2}{2r^2}-\frac{GM}{r}\tag{2.21}$$

The graph of V against r looks like so

- In Newtonian orbits, there is a **centrifugal barrier** $h^2/2r^2$ which dominates as $r\to0$

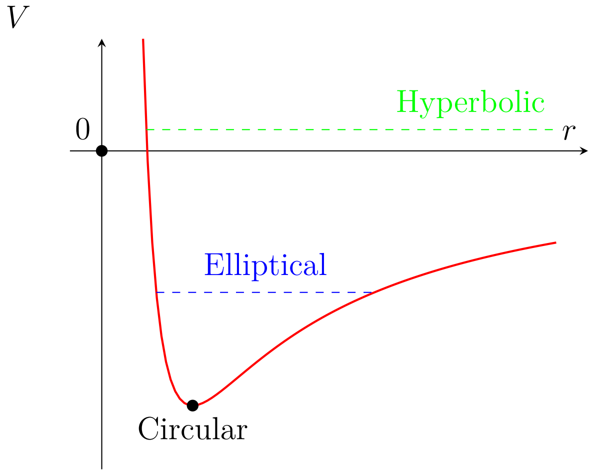

Figure 2.1: Different types of orbits in the Newtonian regime.

- Bound orbits are either circular or elliptical (negative V)
- Unbound orbits have positive V
- Elliptical orbits do not precess (at least if the orbit is isolated, i.e. no perturbations like other planets)

We can (weakly) show the last point by a Taylor expansion of $V(r)$ around $r = r_c$, i.e. for near-elliptical orbits. Indeed, at r_c, the circular orbit radius, all forces are balanced for every position, so the centrifugal barrier and Newtonian potential are equal. Hence $V'(r_c) = 0$. Using this in the Taylor expansion, r_c can be found:

$$V(r) \approx V(r_c) + \frac{1}{2}V''(r_c)(r - r_c)^2. \tag{2.22}$$

Then as the force $\propto -dV/dr$, we will get simple harmonic motion in r with $\boxed{\omega_r^2 = |V''(r_c)|} = GM/r_c^3$. We must find out if this angular frequency really is a precession. Kepler (third law) showed that the *orbital* frequency is equal to the exact same expression. Since there are no additional terms in V to say otherwise, we conclude there is no precession.

General Relativity We now move on to studying GR (Schwarzchild) orbits, which use the full Eq. (2.19). We look at 3 different regimes, shown in Fig. 2.2.

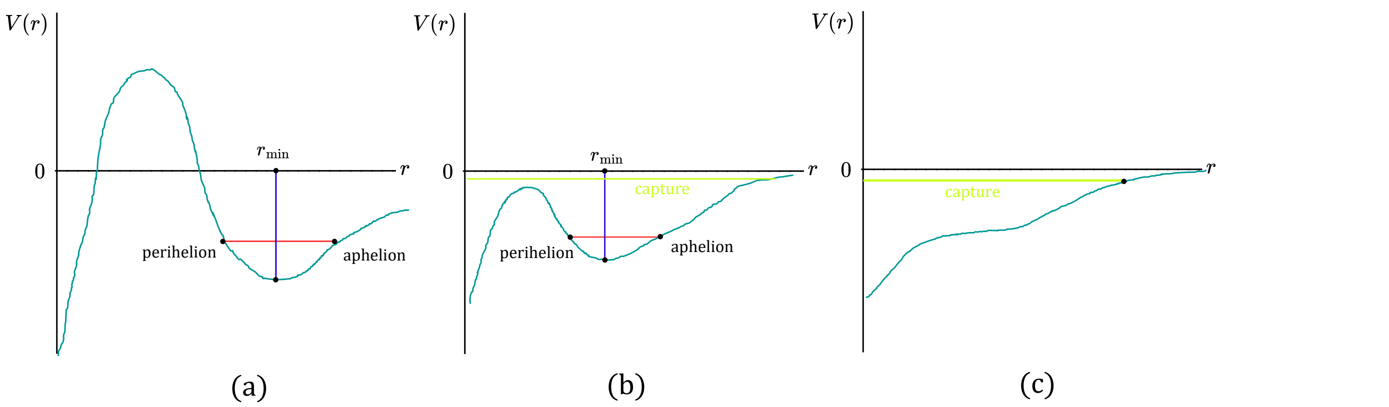

Figure 2.2: Effective potential $V(r)$ against r for (a) large h, (b) intermediate h, and (c) low h. Please do not mind my terrible drawings.

- For large h, we see an attractive regime as $r \to 0$, but this is only for *small* values of r - otherwise basically Newtonian.
- For intermediate h, we still have bound circular orbits and near-elliptical orbits. There are also **capture orbits**, where an objects can be captured and spiral inwards, rather than be deflected.
- New capture orbits are NOT allowed in Newtonian gravity, since there is a V barrier at low r.
- For low h, we *only* have capture orbits.

Remark. Intuitively, this kind of makes sense if you use the description of GR as as a blanket and a heavy ball creates a "dip" in the blanket as in Fig. 2.3. You could kind of

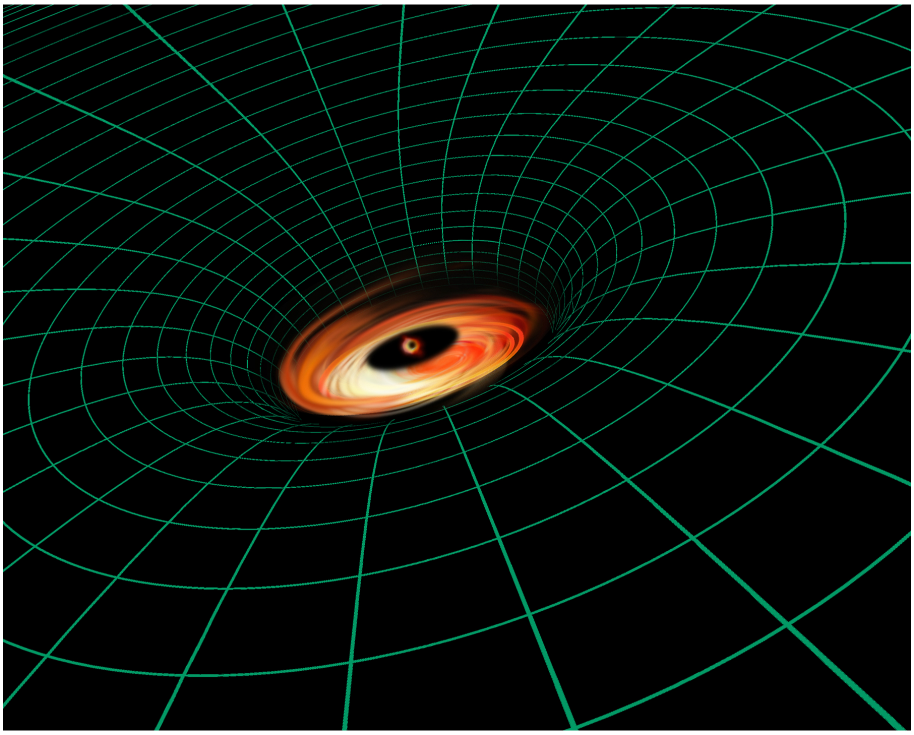

Figure 2.3: Spacetime around a black-hole representation.

see that a slowly-moving object is more likely to 'fall in' than a fast one.

2.3.2 Unstable circular orbits

For intermediate h, we can explicitly find r_c. First, we differentiate V to get

$$V'(r) = -\frac{h^2}{r^3} + \frac{6h^2\mu}{r^4} + \frac{2GM}{r^2} \tag{2.23}$$

Since $r \neq 0$, multiply both sides by r^4, move everything to one side and set it equal to 0, sine we are solving $V'(r_c) = 0$

$$0 = 2\mu c^2 r_c^2 - h^2 r_c + 3h^2\mu \tag{2.24}$$

This is nothing more than the quadratic equation, so we get

$$r_c = \frac{h^2 \pm h\sqrt{h^2 - 12\mu^2 c^2}}{2\mu c^2} \tag{2.25}$$

In Fig. 2.2(b), the stationary points the the maximum and minimum. The $-$ root in Eq. (2.25) corresponds to the **unstable maximum**. The $+$ root exists if $h^2 > 12\mu^2 c^2$, so

$$\boxed{r_c = \frac{h^2}{2\mu c^2} = 6\mu = 3R_\mathrm{s}} \tag{2.26}$$

Inside $r = 3R_\mathrm{s}$, matter drops rapidly into the black hole. Energy released doesn't have time to be thermalised, so it doesn't contribute to accretion radiation.

We can approximate the amount of energy released in accretion. The energy loss from $r = \infty \to 3R_\mathrm{s}$ can be found with $E = kmc^2$. Slow motion in the radial direction allows using Eq. (2.20) with $\dot{r} \sim 0$ for slow motion:

$$\begin{aligned} c^2(k^2-1) = 2V(r) = -\frac{1}{9}c^2 \implies k^2(r_c) = k_c^2 = \frac{8}{9} && (2.27)\\ = \Delta E = |k_c mc^2 - mc^2| \sim 5.7\% \text{ of rest mass} && (2.28) \end{aligned}$$

So accretion onto a non-rotating black hole can potentially radiate almost 6% of rest mass energy. This is almost 10 times more than the 0.6% of energy released from hydrigen to helium fusion!

2.4 Precession of the perihelion of Mercury

In GR, planets move along time-like geodesics of the space-time metric generated by the Sun. We can use the Schwarzchild metric to model this, with the solar mass being much greater than the mass of a planet. We know radial oscillations occur when $\omega_r^2 = V''(r_c)$. First, we start from Eq. (2.24) and rearrange for h^2 to get

$$h^2 = \frac{\mu c^2 r^2}{r - 3\mu} \tag{2.29}$$

We now find V'' explicitly,

$$V''(r) = \frac{3h^2}{r^4} - \frac{12\mu h^2}{r^5} - \frac{2\mu c^2}{r^3}, \tag{2.30}$$

and eliminate h^2 by substituting in Eq. (2.29):

$$\begin{aligned} V''(r) = \omega_r^2 &= \frac{3}{r^4}\left(\frac{\mu c^2 r^2}{r-3\mu}\right) - \frac{12\mu}{r^5}\left(\frac{\mu c^2 r^2}{r-3\mu}\right) - \frac{2\mu c^2}{r^3} && (2.31)\\ &= \left[\frac{3r}{r-3\mu} - \frac{12\mu}{r-3\mu} - 2\right]\frac{\mu c^2}{r^3} && (2.32)\\ &= \frac{r-6\mu}{r-3\mu} \times \frac{\mu c^2}{r^3} && (2.33) \end{aligned}$$

Remark. Taking the limit $\omega_r^2 \to 0$ recovers $r \to 3R_\mathrm{s}$ - this is the **last circular orbit**.

Hence the closest approach to a central mass occurs with a period

$$P_r = \frac{2\pi}{\omega_r} \tag{2.34}$$

When deriving $V(r)$ originally, we made sure to use proper time τ, so P_r is specifically the **period in proper time of the orbiting planet**. During a time P_r, the angle ϕ increases by $\Delta\phi = \dot{\phi}P_r$. In the equatorial plane, $\sin\theta = 1 \implies h = \dot{\phi}r^2$. Therefore,

$$\Delta\phi = \frac{2\pi}{\omega_r}\frac{h}{r^2} \tag{2.35}$$

We will substitute Eq. (2.29) for h^2 and Eq. (2.33) for ω_r^2 into Eq. (2.35). This is just algebra but it is a bit tedious

$$\Delta\phi = 2\pi\left[\frac{\mu c^2 r^2}{r-3\mu}\right]^{1/2} \times r^{-2} \times \left[\frac{r-6\mu}{r-3\mu} \times \frac{\mu c^2}{r^3}\right]^{-1/2} \tag{2.36}$$

Immediately, a lot of things start cancelling out. Pretty much all the constants disappear except for the 2π, and so do any $r - 3\mu$ terms. The r's multiply to leave us with a single r and we arrive at

$$\Delta\phi = 2\pi\left(\frac{r}{r-6\mu}\right)^{1/2}. \tag{2.37}$$

Now, if the planet has precessed by 2π, it is back to its starting state. So to really see if it has precessed, we subtract 2π and we get

$$\Delta\phi = 2\pi\left[\left(\frac{r}{r-6\mu}\right)^{1/2} - 1\right]. \tag{2.38}$$

The formula provided gives the precession in **radians per orbit**. Assuming $r \gg \mu$, which is it for Mercury around the sun (with eccentricity 0.21, semi-major radius of 5.55×10^7 km), we have

$$\delta\phi \sim \frac{6\pi GM}{c^2 r}$$

Now, radians to arcseconds corresponds to multiplying by a factor of 206265 (you don't need to remember that for the exam). One orbit lasts 88 Earth days, or about 0.24 years, and a century is about 36500 Earth days. Taking GM/c^2 to be about 1.47 km, we get $\Delta\Phi \sim 0.103$ arcseconds per century. Now, Mercury's precession is corrected by **43 arcseconds per century!**

2.5 Photon orbits and Gravitational lensing

Photons travel on null geodesics since they are massless. The only change From Eq. (2.17) is that the last equation is 0, i.e.

$$\begin{aligned}
&\left(1-\frac{2\mu}{r^2}\right)t = k \\
&r^2\dot{\phi} = h \\
&c^2\left(1-\frac{2\mu}{r}\right)t^2 - \left(1-\frac{2\mu}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 = 0,
\end{aligned} \tag{2.39}$$

and that this is the *only* change. The new ODE we get is

$$\dot{r}^2 + \frac{h^2}{r^2}\left(1-\frac{2\mu}{r}\right) = c^2 k, \tag{2.40}$$

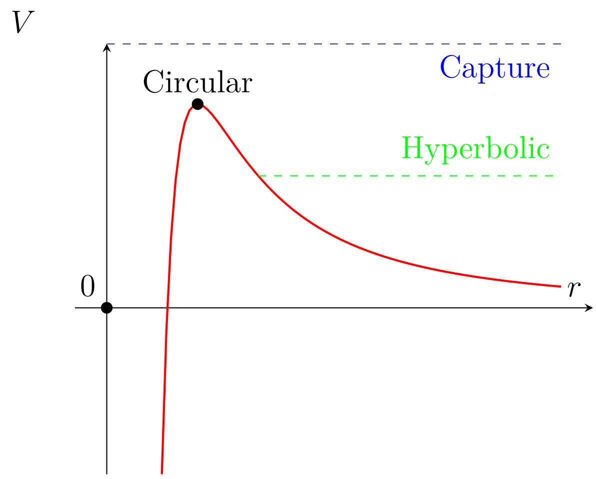

Figure 2.4: Different types of photon orbits.

which produces an effective potential

$$V(r) = \frac{h^2}{2r^2}\left(1 - \frac{2GM}{rc^2}\right), \tag{2.41}$$

so we do not have the Newtonian part like in Eq. (2.19) but keep the GR part. A plot of this is shown in Fig. 2.4. There are capture orbits, where like in the massive case, photons are swept around by the curvature of the manifold but do not keep in an orbit. They can also form circular orbits, **but never elliptical orbits**. It is either circular or they get flung out of orbit (hyperbolic).

2.5.1 Gravitational lensing

Orbits with $r \gg \mu = GM/c^2$ have small angle deflections. Using GR, we will calculate this deflection. We will use the chain rule to eliminate $\dot{r}$ from Eq. (2.40) as follows:

$$\begin{aligned} \dot{r} = \frac{dr}{d\tau} = \frac{dr}{d\phi}\frac{d\phi}{d\tau} = \dot{\phi}\frac{dr}{d\phi} = \frac{h}{r^2}\frac{dr}{d\phi} \\ \frac{h^2}{r^4}\left(\frac{dr}{d\phi}\right)^2 + \frac{h^2}{r^2}(1-2\mu) = c^2k. \end{aligned} \tag{2.42}$$

We will now make a change of variables to $u = 1/r$. This is purely algebra. We also divide by h^2 and start from there:

$$\frac{1}{r^4}\left(\frac{dr}{d\phi}\right)^2 + \frac{1}{r^2}(1-2\mu) = \frac{c^2k}{h^2} \tag{2.43}$$

$$u^4\left(\frac{d(1/u)}{d\phi}\right)^2 + u^2 - 2\mu u^3 = \frac{c^2k}{h^2} \tag{2.44}$$

$$u^4\left(\frac{du}{d\phi} \times -\frac{1}{u^2}\right)^2 + u^2 - 2\mu u^3 = \frac{c^2k}{h^2} \tag{2.45}$$

$$\left(\frac{du}{d\phi}\right)^2 + u^2 - 2\mu u^3 = \frac{c^2k}{h^2} \tag{2.46}$$

The trajectory of the light ray can be determined directly from this equation, much as we did for the precession, but the calculation requires some care. Therefore, and for variety, we take a different tack by differentiating to arrive at the second order ODE

$$\frac{d^2u}{d\phi^2} + u - 3\mu u^2 = 0. \tag{2.47}$$

To solve this equation we adopt a perturbative approach. We assume $r \gg \mu$ and write u as a power series: $u = u^{(0)} + u^{(1)} + \ldots$, with $u^2 \ll u$ assumed to allow for convergence of the power series. We can then solve Eq. (2.47) for each power:

$$\begin{aligned}
\text{zeroth order} \quad \frac{d^2u^{(0)}}{d\phi^2} + u^{(0)} &= 0, \\
\text{first order} \quad \frac{d^2u^{(1)}}{d\phi^2} + u^{(1)} &= 3\mu \left(u^{(0)}\right)^2, \\
\ldots\cdots &= \cdots
\end{aligned} \tag{2.48}$$

At order zero the solution is $u^{(0)} = a \sin\phi$ and corresponds to the straight line $y = 1/a$. The constant of integration a is the reciprocal distance of closest approach of the light ray to the central mass[1]. Then, at first order we find

$$\frac{d^2u^{(1)}}{d\phi^2} + u^{(1)} = \frac{3\mu}{2b^2}(1 - \cos 2\phi), \qquad \Longrightarrow \qquad u^{(1)} = \frac{3\mu a^2}{2}\left(1 + \frac{1}{3}\cos 2\phi\right)$$

Thus the photon trajectory, incorporating the first order correction coming from general relativity, is described by

$$u = \frac{1}{r} = a\sin\phi + \frac{3\mu a^2}{2}\left(1 + \frac{1}{3}\cos 2\phi\right) + \cdots$$

We now determine how this affects the asymptotic directions of the light ray by considering the limiting values of the angle ϕ as $u \to 0$ ($r \to \infty$). Retaining only the zeroth order solution these directions are $\phi_- = 0$ and $\phi_+ = \pi$; adding the first order correction (we assume ϕ small and expand $\cos(2\phi)$) these become

$$a\sin\phi + 2\mu a^2 = 0, \quad \Rightarrow \quad \phi_- = -2\mu a \text{ and } \phi_+ = \pi + 2\mu a$$

In particular,

Deflection of light

$$\Delta\phi \sim 4\mu a = \frac{4GM}{c^2 r_0} \tag{2.49}$$

In 1919 Arthur Eddington made observations of the positions of stars close to the sun during a total solar eclipse and compared these to their known positions from when their starlight does not pass so close to the sun. The solar radius is approximately 6.96×10^5 km and $M = 1.48 M_\odot$ so that the prediction for the defection of the starlight is 8.5×10^{-6} radians.

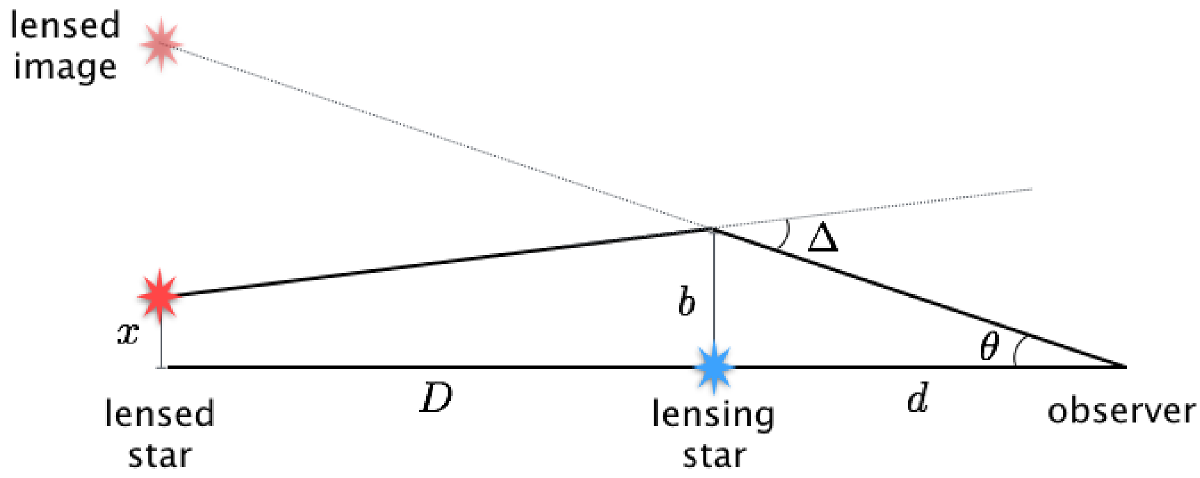

Figure 2.5: Schematic of how Einstein rings are formulated.

2.5.2 Einstein rings

The idea is shown in Fig. (2.5), where a 'lens' a distance d away deflects the light coming from a source a distance D beyond it and displaced off the direct line of sight by a distance x. Doing some small angle geometry:

$$\Delta = \frac{4\mu}{b} \simeq \frac{b}{d} + \frac{b-x}{D} \tag{2.50}$$

Solving for the observation angle $\theta \approx b/d$ gives

$$\theta = \frac{x}{2(D+d)} \pm \left[\frac{4\mu D}{d(D+d)} + \left(\frac{x}{2(D+d)}\right)^2\right]^{1/2} \tag{2.51}$$

Why are there two values? The light we see is travelling along a null geodesic and geodesics are critical points of the distance (or time) function. The path shown in Fig. 2.5 where the light arrives to us by going above the lens is a critical point; so too is a path that passes below the lens, which is the other solution. As the lens and source become closer aligned, that is, $x \to 0$, the 2 light rays (above and below) also get brought closer together. At a sufficient small x, all paths around the lensing star become critical points and the imaged star becomes a perfect ring. The apparent opening angle of such an Einstein ring is

$$\theta = \left(\frac{4\mu D}{d(D+d)}\right)^{1/2} \tag{2.52}$$

2.6 Schwarzchild black holes

Recall the third equation of Eq. (2.17) when we were first discussing R_s. We now aim to study how a close and far observer see the black hole. We rearrange the third equation of Eq. (2.17)

$$\frac{1}{c}\frac{dr}{d\tau} = \pm\left(\kappa^2 - 1 + \frac{2\mu}{r}\right)^{1/2}, \tag{2.53}$$

[1]Astute (or brainrotted?) physicists will recognise $1/a$ as the impact parameter

where $+$ is for outfall (away) and $-$ for infall (towards). We suppose the observer falls from $R \to r$ with the position at R assigned the initial time $\tau = 0$. Then

$$c\tau = \int_r^R \frac{dr}{\left(\kappa^2 - 1 + \frac{2\mu}{r'}\right)^{1/2}}, \tag{2.54}$$

where κ is a constant of motion which is γ^2 as $r \to \infty$. This integral is **not singular at** $r = 2\mu$, so the **observer falls through the event horizon in finite proper time**. A person falling freely inwards, approaching the horizon radially, will pass straight through it; it is not a singularity of the space-time.

We now do the same for a *distant observer*. We use the chain rule so that

$$\frac{dr}{d\tau} = \frac{dr}{dt}\frac{dt}{d\tau}$$

$$\begin{aligned} \frac{1}{1-\frac{2\mu}{r}}\kappa^2 - \frac{1}{\left(1-\frac{2\mu}{r}\right)^3}\frac{\kappa^2}{c^2}\left(\frac{dr}{dt}\right)^2 &= 1, \\ \Longrightarrow \frac{\kappa}{c}\frac{dr}{dt} &= -\left(1 - \frac{2\mu}{r}\right)\left|\kappa^2 - 1 + \frac{2\mu}{r}\right|^{1/2}, \end{aligned} \tag{2.55}$$

Then

$$cT = \int_r^R \frac{\kappa dr'}{(1 - 2\mu/r)\left(\kappa^2 - 1 + 2\mu/r'\right)^{1/2}} \tag{2.56}$$

for the time that we record it takes for the observer to get to the radial distance r. T now diverges at $r = R_{\mathrm{s}}$, so

A distant observer can never see an object hit the event horizon.

It implies that we can never receive *any* information that such an observer obtains from the event horizon, or from the region of space-time inside it.

Light travels along null geodesics, we can try to find its trajectory from $r \to R$

$$\left(1 - \frac{2\mu}{r}\right)c^2 dt^2 = \frac{1}{1-\frac{2\mu}{r}}dr^2, \tag{2.57}$$

so that the time we measure for it to reach us is

$$cT = \int_r^R \frac{dr'}{1 - \frac{2\mu}{r'}} = R - r + 2\mu \ln\left|\frac{R-2\mu}{r-2\mu}\right|. \tag{2.58}$$

Notice $t \to \infty$ as $r \to 2\mu$, so **there is no information from** $r \leq R_{\mathrm{s}}$, and you are stuck only being able to obtain information outside of the event horizon.

2.6.1 Gravitational time dilation

In Eq. (2.58), $T = t_{\mathrm{B}} - t_{\mathrm{A}}$ where

- t_{A} is the coordinate time light leaves r
- t_{B} is the coordinate time light reaches R

However, the RHS of Eq. (2.58) is *not* time-dependent, so

Pulses of light from $r \to R$ always take the same coordinate time $T = \Delta t$.

So an emitter and observer have the same time between pulses/observations $T = \Delta t$.

Now, we know $ds^2 = -c^2 d\tau^2$ so

$$c^2(\Delta\tau)^2 = c^2\left(1 - \frac{2\mu}{r}\right)dt^2$$

Now for a distant observer (the destination), the time difference in the observer's frame is the proper time difference and equals $\Delta\tau_{\mathrm{B}} = \Delta t$. Now, we can backtrace to r:

$$\Delta\tau_{\mathrm{A}} = (1 - 2\mu)\Delta t = \left(1 - \frac{2\mu}{r}\right)\Delta\tau_B < \Delta\tau_B \tag{2.59}$$

The observer at R must conclude time at $r < R$ **moves more slowly for them** - we have just shown gravitational time dilation!

2.6.2 Gravitational redshift

This is a problem sheet question and came up in a previous exam paper, although by a different lecturer. Suppose you are Roy, and you have decided to find your local black hole and venture towards its event horizon. He will ensure to send back signals to us plebs back on Earth. However, these signals are coming from a region of strong gravitational field and are perceived by us, in a much weaker region of the gravitational field, as being greatly **redshifted**.

To find the 4-wavevector k_μ, we need satisfy some criteria

- $k_\mu k^\mu = 0$
- As $r \to \infty$, the 4-wavevector is the same as SR, that is $k_\mu = (-k_t, k_x, k_y, k_z)$ for some values $k_t, k_x, k_y, k_z \in \mathbb{R}$.
- Conserves energy (we require this, and have to be careful since we also have gravitational time dilation).

We will skip over the derivation, though eigenchris's video goes through it. You may take me on oath if you wish, but the 4-wavevector (or more specifically, the 1-form) is

$$k_\mu = \left(-k_\infty, \frac{k_\infty}{1 - \frac{2\mu}{r}}, 0, 0\right) \tag{2.60}$$

where k_∞ is the wavenumber at $r = \infty$.

Now, Roy is in his own frame, and we, the observers are at $r = r_A$, we measure his frequency as:

$$\omega_A = \frac{\omega_\infty}{\left(1 - \frac{2\mu}{r_A}\right)^{1/2}}, \quad \text{or} \quad \omega_A\left(1 - \frac{2\mu}{r_A}\right)^{1/2} = \omega_\infty. \tag{2.61}$$

If Roy always transmits at a fixed frequency as he perceives it (fixed value of ω_A) then as he gets closer and closer to the event horizon the signal appears in the asymptotic far field to be redder and redder, eventually being so close that $\omega_\infty \to 0$.

2.6.3 Shapiro delay

This is another test of GR based on the time taken to send radar signals to the inner planets (Venus) and back. this is useful as another test of GR and appears in Problem Sheet 7 and past papers.

The travel time for the signal depends on whether Venus is at the point closest to ourselves in its orbit or the point furthest away or somewhere in between. In addition, when it is furthest away, the signal passes through the strong gravitational field close to the Sun and is affected by it; this gives an extra contribution to the time taken and is the delay calculated, and measured, by Shapiro.

Theorem 2.6.1. *Shapiro delay. The time taken, T, for a signal to travel between the point of its closest approach to the Sun, at distance $r = b$, and a general point at $r = R$.*

$$cT = \sqrt{R^2 - b^2} + 2\mu \left[\ln \frac{R + \sqrt{R^2 - b^2}}{b} + \frac{1}{2} \left(\frac{R-b}{R+b} \right)^{1/2} \right] + \cdots . \tag{2.62}$$

Proof. The proof (with steps) is shown in 2019 Q 2c,d,e. See Section 5.3.1 for this. □

2.7 Kruskal-Szekeres Coordinates

The Schwarzchild metric we have used so far have proven to be abundantly useful. However, the main issue we have encountered is the event horizon, where we cannot obtain any information beyond it. If you were to draw the worldline of light, we get an asymptote at $r = R_s$: As $r \to \infty$ gradient $\to \pm 1$ and the light cone has a $45°$ opening angle, Fig 2.6. For $r \to 2\mu = R_s$, the gradient of the worldline $\to \pm\infty$. World Lines at $R_s \pm \delta$ meet at ∞ as $\delta \to 0$. The line at $r = R_s$ is actually a point at $r = R_s; t = \infty$. This is clearly **not** a good coordinate system to study black holes.

The Kruskal-Szekeres (KS) coordinates aim to solve this. They are defined as

Definition 2.7.1. KS-coordinate transform on the Schwarzchild metric.

$$r > 2\mu : \qquad u = \alpha \cosh \beta \qquad v = \alpha \sinh \beta \tag{2.63}$$

$$r < 2\mu : \qquad u = \alpha \sinh \beta \qquad v = \alpha \cosh \beta \tag{2.64}$$

where Kruskal's choice of α, β were

$$\alpha = \left| 1 - \frac{r}{2\mu} \right|^{1/2} \qquad \beta = \frac{ct}{4} 4\mu. \tag{2.65}$$

The metric now becomes

$$ds^2 = -32 \frac{\mu^3}{r} e^{-\frac{r}{2\mu}} \left(dv^2 - du^2 \right) + r^2 d\Omega^2. \tag{2.66}$$

Here, r is still the Schwarzchild r coordinate, but is no longer independent since $r = r(u, v)$.

From this metric, we can derive some interesting properties

1. From $\cosh^2 x - \sinh^2 x = 1$,

$$\left(\frac{r}{2\mu} - 1 \right) e^{\frac{r}{2\mu}} = u^2 - v^2$$

The RHS is a **hyperbola**, so for fixed $r, (u, v)$ curves are **hyperbolae**.

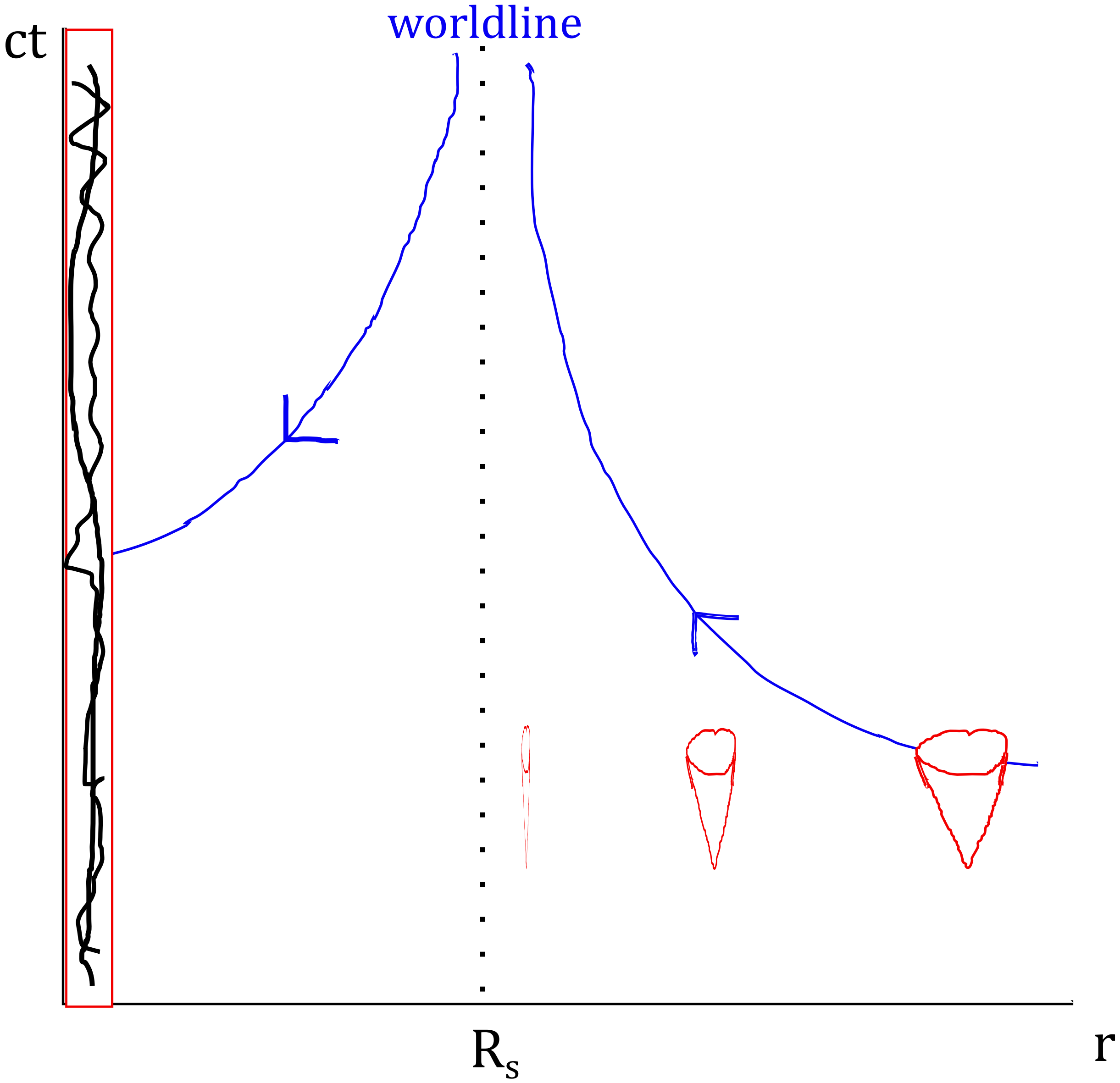

Figure 2.6: Wordline (blue) of light near a black hole and its event horizon. The red cones are light cones positioned at different values of r.

2. Light infall for $d\theta = d\phi = 0$, along null geodesics $ds = 0$. Then $dv = \pm du$, i.e. the opening angle of light cones is constant.

3. Lines of constant t are straight lines radially out from $u = v = 0$

We can draw out the result of this and it is shown in Fig. 2.7.

There are a few points to go through:

- The KS diagram is **symmetric**.
- Region 1 is the universe outside $r = R_s$.
- Region 2 is the universe inside $r = R_s$. It is a **null surface**; even light emitted from within region II cannot escape it and necessarily arrives as we discovered in Section 2.6.
- Region III is a time-reversed copy of our universe. It contains a singularity at $u^2 - v^2 = -1$ with $t < 0$.

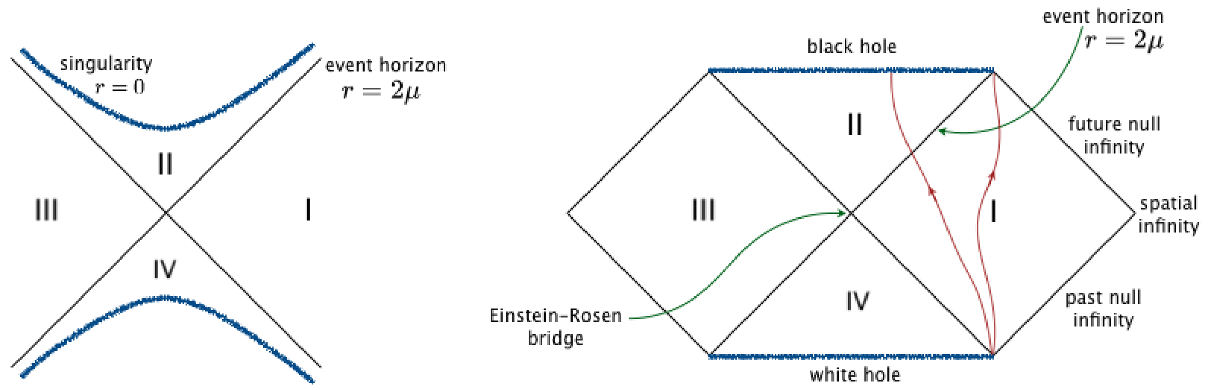

Figure 2.7: (left) Drawing of Kruskal-Szekeres coordinates, and (right) Penrose diagram. Null directions are preserved to allow correct causal structure. Two time-like trajectories are indicated on the RHS, showing the event horizon can be passed through. Diagram modified from Dr. Gareth Alexander's previous lecture notes.

- Region IV is a **white hole**. Any light signal emitted from within region IV will cross the event horizon and leave it.

2.8 Kerr black hole

The Kerr metric is a generalisation for an axisymmetric rotating black hole, that was done on top of a generalisation to the KS metric to account for *charged* black holes (Reissner-Nordström BH). Without derivation, the metric is

$$ds^2 = -c^2 \left(\frac{\Delta - a^2 \sin^2\theta}{\rho^2}\right) dt^2 - 2ac\frac{2\mu r \sin^2\theta}{\rho^2} dt d\phi + \frac{(r^2 + a^2) - a^2\Delta \sin^2\theta}{\rho^2} \sin^2\theta d\phi^2$$
$$+ \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 \tag{2.67}$$

where

- $\Delta = r^2 - 2\mu r + a^2$
- $\rho^2 = r^2 + a^2 \cos^2\theta$
- $a = J/\mu$ where J is the total angular momentum.

The coordinates are similar to Schwarzchild but

- t is a frame in which everything is stationary.
- r is not defined from the circumference of a circle as there is no $r^2 d\Omega^2$ term in Eq. (2.67).

Then ds^2 is

$$ds^2 = g_{tt} c^2 dt^2 + 2g_{t\phi} dt d\phi + g_{\phi\phi} d\phi^2 + g_{rr} dr^2 + g_{\theta\theta} d\theta^2 \tag{2.68}$$

Remark. Eq. (2.67) is still a metric because it is symmetric. Since we have a term $dtd\phi$, this means $g_{t\phi} = g_{\phi t}$ so we end up double counting in ds^2.

2.8.1 Ergosphere

Just like in the Schwarzchild case, there is a surface where $g_{rr} \to \infty$ which happens when

$$\Delta = r^2 - 2\mu r + a^2 = 0$$

Whether there is a real solution or not depends on the magnitude of a. Indeed, for $|a| < \mu$, there are 2 real solutions:

$$r_{\pm} = \mu \pm \left(\mu^2 - a^2\right)^{1/2} \tag{2.69}$$

- The positive solution r_+ is the **event horizon** of the black hole; massive, or massless, particles can cross it falling inwards, but can then never cross the same surface again to get back out.
- the negative solution r_- is the **Cauchy horizon**. It is called that because after this point, we cannot determine the worldlines from initial values (named after the mathematician).

However, if $|a| > \mu$, we have no real roots and therefore no horizon. Hence, there is no horizon 'hiding' the singularity - we call this a **naked singularity**.
Theorem 2.8.1. *Cosmic censorship conjecture (not really a true theorem, sorry mathematicians). There are no naked singularities. Hence $|a| \leq \mu$.*

Unlike in the Schwarzchild case, g_{tt} vanishes on a different surface in the Kerr metric, whilst in the former, g_{tt} and g_{rr} did their shenanigans on the same surface.

We suppose a photon is emitted tangent to the $\theta = \pi/2$ plane, the equatorial plane as before. Photons travel on null geodesics, so $ds^2 = 0$. Since the photon travels around the equator, we know $dt, d\phi$ are non-zero and keep those terms:

$$ds^2 = 0 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2. \tag{2.70}$$

We can treat Eq. (2.70) as a quadratic equation in $d\phi$ and use the quadratic formula in this sense to get

$$\frac{d\phi}{dt} = -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \left(\left(\frac{g_{t\phi}}{g_{\phi_\phi}}\right)^2 - \frac{g_{tt}}{g_{\phi\phi}}\right)^{1/2} \tag{2.71}$$

If $g_{tt} = 0$, we get 2 solutions for $d\phi/dt$:

$$\frac{d\phi}{dt} = -2\frac{g_{t\phi}}{g_{\phi\phi}} \qquad \text{OR} \qquad \frac{d\phi}{dt} = 0$$

This gives the key result

- Photons launched in the direction of motion move in that direction - this is **frame dragging**.
- Photons launched in the *opposite* direction of motion are stationary.

Definition 2.8.1. The region enclosed by the $g_{rr} = \infty$ and $g_{tt} = 0$ surfaces is the **ergosphere**. A typical ergosphere is shown in Fig. 2.8.

We can also look at the Penrose diagram for this (non-examinable).

- On passing inward through the inner horizon, the infaller sees the infinite past of the Universe reflected in the gravitationally repulsive singularity;

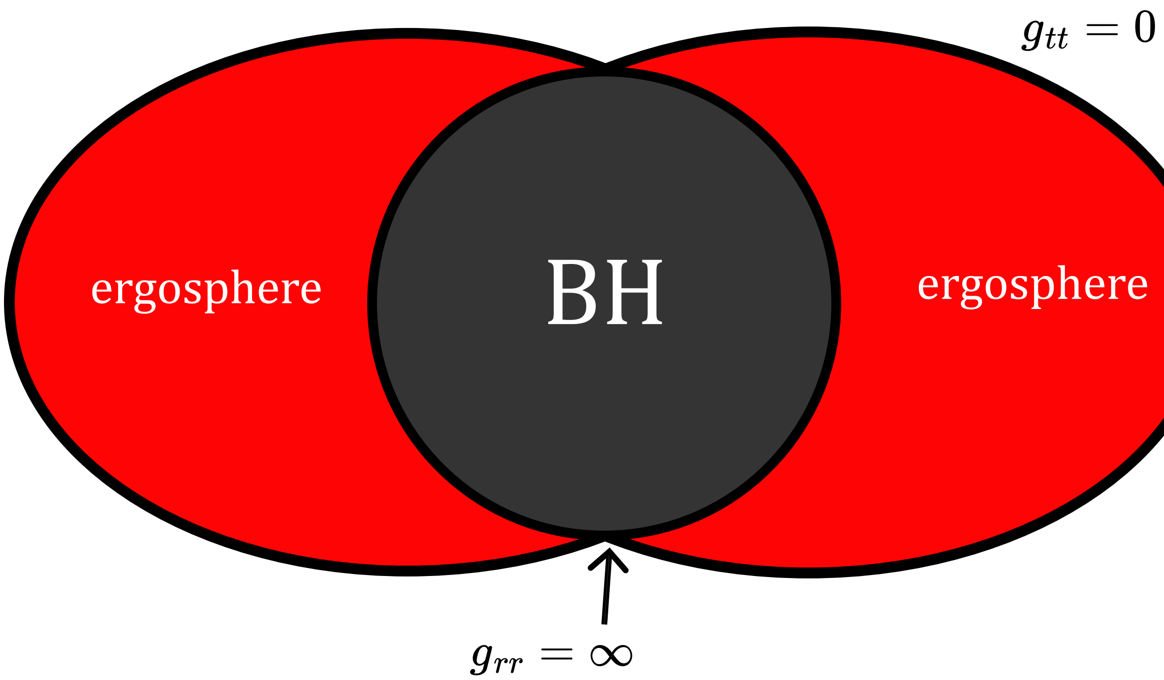

Figure 2.8: Diagram of the ergosphere of a Kerr black hole.

- On passing back outward through the inner horizon, the infaller sees the infinite future of the Universe;
- On passing outward through the outer horizon of the white hole, the infaller sees the infinite past of the New Universe.
- It is possible to pass through the disk bounded by the ring singularity of the rotating black hole to an antiverse.

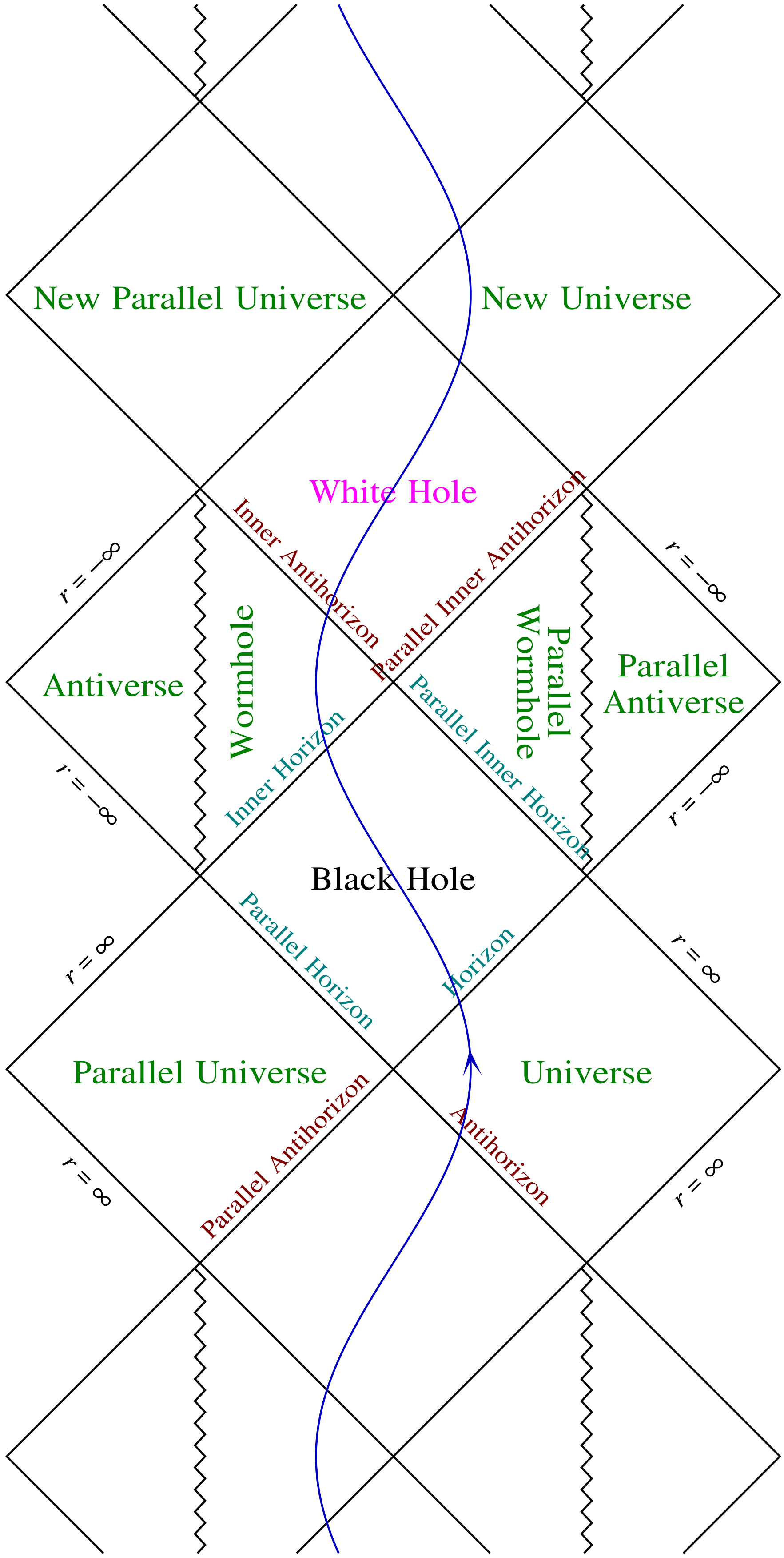

Figure 2.9: Penrose diagram for Kerr black hole.

Chapter 3

Cosmology

3.1 Modelling the universe

Observations about the universe:

- On scales > 10Mpc the universe is **homogeneous**
- Universe is isotropic [1 pc $= 3 \times 10^{16}$ m; Galaxies about 15kpc; galactic clusters are about a Mpc away.
- Most distant galaxies ≥ 10 GPc away.
- Fluctuations in the Cosmic Microwave Background (CMB) about 1 in 10^5.
- Galaxies receeding at a speed $v = Hd$.
- Average density 10^{-26} kg m $= \rho$

Expect GR to be needed when $R \sim \frac{GM}{c^2}$

$$R = \frac{GM}{c^2} = \frac{G}{c^2}\frac{4}{3}\pi R^3 \rho \implies R \sim 6 \text{ Gpc}$$

So the large structure of the universe requires GR to be described. To do so, we must establish a good coordinate system.

Definition 3.1.1. Cosmic time is the time coordinate for the same synchronous time *everywhere*. It is the **local proper time** and must be **stationary w.r.t CMB**.

In general, the metric satisfies $ds^2 = -c^2dt^2 + dl^2$ where dl^2 is the space component. For a fixed position in time, we have $dl^2 = 0$ so $ds^2 = -c^2 f\tau^2 = -c^2dt^2 \implies dt = d\tau$.

The space component dl^2 satisfies **constant curvature everywhere**. This type of universe is called a **de-Sitter universe**, denoted dS_4 and is an exact solution to Einstein's equations. It is defined as the subset of $\mathbb{R}^{1,4}$ (that is, the space of vectors (x^0, x^1, x^2, x^3)) satisfying

$$-(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = -a_0^2$$

where $a_0 \in \mathbb{R}$ is a space-like distance from some origin.

3.2 Friedmann-Lemaître-Robertson-Walker Metric

de-Sitter universes are isotropic and homogeneous, matching observations. Therefore, the general metric takes the form

$$ds^2 = -c^2dt^2 + a^2(t)dl^2 \tag{3.1}$$

where dl^2 is the metric for an isotropic, homogeneous, three-dimensional Riemannian manifold[1] and $a : \mathbb{R} \to \mathbb{R}$ is a continuous function of cosmic time. The most typical surfaces of constant curvature are n-spheres. A 2-sphere obeys the equation

$$x^2 + y^2 + z^2 = R^2,$$

where $(x, y, z) \in \mathbb{R}^3$ and $x, y, z, R \in \mathbb{R}$. So a 2-sphere is a 2D surface *embedded* in a three-dimensional space (at least, that is our visualisation, remember that if you're on a manifold, you do *not* have an embedding space). We can generalise this to a 3-sphere. Define $\rho^2 = x^2 + y^2 + z^2$, and introduce a new space dimension ω. Then

$$\rho^2 + \omega^2 = a^2 \in \mathbb{R} \tag{3.2}$$

Respecting the d on both sides, we have $\rho d\rho = -\omega d\omega$. Thus dl^2 becomes

$$dl^2 = \frac{d\rho^2}{(1 - \rho^2/a^2)} + \rho^2 d\Omega^2 \tag{3.3}$$

The radial coordinate r will then be defined as $\rho = ar$ and we get

Friedmann-Lemaître-Robertson-Walker (FLRW) Metric

$$ds^2 = -c^2dt^2 + a^2(t)dl^2 = -c^2dt^2 + a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2\right], \tag{3.4}$$

where $k = 0, -1, +1$ corresponding to the cases

$$dl^2 = \begin{cases} d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta d\phi^2\right) & k = +1,\ r = \sin\chi \\ d\chi^2 + \chi^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) & k = 0 \\ d\chi^2 + \sinh^2\chi\left(d\theta^2 + \sin^2\theta d\phi^2\right) & k = -1,\ r = \sinh\chi \end{cases} \tag{3.5}$$

and thus FLRW space-time is expressed in the coordinates (t, χ, θ, ϕ).

- $k = +1$, metric of a 3-sphere with positive curvature $1/a^2$,
- $k = 0$, zero scalar curvature - flat spacetime
- $k = -1$, negative scalar curvature - open space (also called hyperbolic space).

Here, χ is sometimes referred to as the **conformal distance** or **cosmic radius**.

3.2.1 Hubble's law

The FLRW metric treats the universe as a fluid which is homogeneous and isotropic. This cosmic fluid is at rest in the co-moving (t, χ, θ, ϕ) coordinate system , where t is the proper time. At any choice of t at any spatial point, the distribution of matter/energy/speeds look the same.

[1] This is because we expect metrics in pure space to still be something like $dx^2 + dy^2 + dz^2$, which is Riemannian. The entire manifold (space and time) in a de-Sitter universe is still pseudo-Riemannian.

As a result, some galaxies in the FLRW metric are stationary, so the distance is the proper distance d_p

$$d_p = \int_0^{\chi} a(t) d\chi' = a(t)\chi \tag{3.6}$$

Proper distance is therefore measured with $dt = 0$ and $ds = a d\chi$, because we assume that light from the galaxy is travelling along a radial null geodesic. Now, χ is fixed, so

$$v = \frac{d}{dt}(a\chi) = \frac{da}{dt}\chi = \dot{a}\chi, \tag{3.7}$$

However $\chi = d_p/a$ so

$$v = H(t)d_p = \frac{\dot{a}}{a}d_p \tag{3.8}$$

This is **Hubble's law**. If $t = t_0$ which is today, then $H(t_0) = H_0$ - Hubble's 'constant', which changes with time.

3.2.2 Redshift

Now consider a galaxy also at χ, which emits photons at time t_e and $t_e + \delta t_e$. The observer, let's say Oscar, observes these pulses at t_o and $t_o + \delta t_o$. As usual, light travels along null geodesics $ds^2 = 0$ so

$$c dt = a(t) d\chi \tag{3.9}$$

$$\chi = \int_{t_e}^{t_o} \frac{c}{a(t)} dt = \int_{t_e + \delta t_e}^{t_o + \delta t_o} \frac{c}{a(t)} dt \tag{3.10}$$

Now, since $a(t)$ is continuous, it is integrable. We assume $t_e < t_e + \delta t_e < t_o < t_o + \delta t_o$ and we can subtract the integral $\int_{t_e+\delta t_e}^{t_o} c/a(t) dt$ from both sides and rearrange:

$$\int_{t_e}^{t_o} \frac{c}{a(t)} dt - \int_{t_e+\delta t_e}^{t_o} \frac{c}{a(t)} dt = \int_{t_e+\delta t_e}^{t_o+\delta t_o} \frac{c}{a(t)} dt - \int_{t_e+\delta t_e}^{t_o} \frac{c}{a(t)} dt \tag{3.11}$$

$$\implies \int_{t_e}^{t_e+\delta t_e} \frac{c}{a(t)} dt = \int_{t_o}^{t_o+\delta t_o} \frac{c}{a(t)} dt \tag{3.12}$$

$$= \int_{\delta t_e}^{\delta t_o} \frac{c}{a(t)} dt \tag{3.13}$$

$$\implies \frac{\delta t_e}{a(t_e)} = \frac{\delta t_o}{a(t_o)} \tag{3.14}$$

We can rearrange this expression such that

$$\frac{\delta t_o}{\delta t_e} = \frac{a(t_o)}{a(t_e)} \tag{3.15}$$

Now, the angular frequency of the light is defined as $\omega \propto 1/\lambda$, hence Eq. (3.15) is also the same as

Definition 3.2.1. The **redshift** z[2] is defined as

$$\frac{\omega_e}{\omega_o} = \frac{a(t_o)}{a(t_e)} := 1 + z \tag{3.16}$$

It is the factor by which the universe **expands** between emission and observation.

[2]The derivation is not strictly correct because the light travel distance of the emitted photons are the same.

Note that redshift and Hubble's law come from the same idea - consider a fixed galaxy and a light signal. Thus, we can actually derive Hubble's law from this redshift consideration.

If the galaxies are close to each other, so Oscar's times are $t_{\text{emit}} = t_{\text{obs}} - \delta t$ with δt small, then we may approximate

$$\frac{a\left(t_{\text{obs}}\right)}{a\left(t_{\text{emit}}\right)} \approx \frac{a\left(t_{\text{obs}}\right)}{a\left(t_{\text{obs}}\right) - \partial_t a\left(t_{\text{obs}}\right) \cdot \delta t} \approx 1 + \left.\frac{\dot{a}}{a}\right|_{t_{\text{obs}}} \delta t \approx 1 + H\left(t_{\text{obs}}\right) \frac{a\left(t_{\text{obs}}\right) \chi_0}{c}$$

where $H_(t_{\text{obs}}) = H(t_o) = H(t_0)$ is today's value of Hubble's constant. For small redshifts, the interpretation as a Doppler shift due to the apparent recession velocity gives $z = v/c$ and we obtain Hubble's law

$$v = H(t_0) d_p,$$

as before in Eq. (3.8). We can thus interpret $1 + z$ as the cosmological redshift showing how much local gravity (or galaxy speeds) in (χ, θ, ϕ) will contribute to a closed universe.

3.3 Solving the field equations

The FLRW metric gives $g_{\alpha\beta}$ in (t, χ, θ, ϕ) space. We can therefore find $\Gamma^{\alpha}_{\gamma\beta}$, $R_{\alpha\beta}$ and R. You can do it yourself because I also cannot be asked to type this out and we get

$$\begin{aligned} R_{00} - \frac{1}{2} R g_{00} &= \frac{3}{c^2} \left[\left(\frac{\dot{a}}{a} \right)^2 + \frac{kc^2}{a^2} \right], \\ R_{ij} - \frac{1}{2} R g_{ij} &= \frac{-1}{c^2} \left[\frac{2\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 + \frac{kc^2}{a^2} \right] g_{ij}, \; i,j = 1,2,3 \end{aligned} \tag{3.17}$$

Now we have the LHS sorted, we need the RHS - that is $T_{\alpha\beta}$ using Eq. (1.58) in a cosmic time, comoving frame. Since we are co-moving, the 4-velocity components $v^1 = v^2 = v^3 = 0$ and $g_{tt} v^t v^t = -c^2$. With $g_{tt} = -1$, $v^t = c$

$$\begin{aligned} T_{00} &= \left(\rho + \frac{p}{c^2} \right) c^2 - P = \rho c^2 \\ T_{ij} &= P g_{ij}, \; i,j = 1,2,3 \end{aligned} \tag{3.18}$$

Substituting everything into the field equations, we get 4 equations collectively called the **Friedmann equations**

Friedmann equations

The 00-component is called the **Friedmann equation**

$$\left(\frac{\dot{a}}{a} \right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3} \rho \tag{3.19}$$

The last 3 equations make up the **acceleration equation**

$$2 \frac{\ddot{a}}{a} = -\frac{8\pi G}{3c^2} \left(\rho c^2 + 3P \right) \tag{3.20}$$

The RHS of the acceleration equation is **negative**, so the universe **cannot be static**. Observations show that $\dot{a} > 0$, i.e. the universe is expanding, but since $\ddot{a} < 0$, the universe must have been smaller in the past. We can therefore perform a cheap extrapolation of time

$$\frac{1}{H} = \frac{a}{\dot{a}}\bigg|_{\text{now}} \sim 14 \text{ billion years} \tag{3.21}$$

The Friedmann equations suggest $a \to 0$, and back in time, the universe must have been vanishingly smal - the idea of a 'priemevial atom' from Lemaître and the 'Big Bang' from Hoyle.

3.3.1 Cosmological constant Λ

We saw that with the Friedmann equations, the universe was not static. Einstein did not like this and modified his equations to mathc his belief that the universe was static, by way of the **cosmological constant** Λ:

$$R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta} + \Lambda g^{\alpha\beta} = kT^{\alpha\beta}. \tag{3.22}$$

It is common to rearrange the Λ term to the RHS and rewrite it as

$$R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta} = k\left(T^{\alpha\beta} - \frac{\Delta}{k}g^{\alpha\beta}\right), \tag{3.23}$$

i.e., this term acts as some new fluid satisfying the energy-momentum tensor as

$$\left(\rho_\lambda + \frac{P_\lambda}{c^2}\right)v^\alpha v^\beta + P_\lambda g^{\alpha\beta}, \tag{3.24}$$

which is satisfied if

$$\rho_\Lambda = \frac{\Lambda c^2}{8\pi G} \quad P_\Lambda = -\frac{\Lambda}{k} = -\frac{\Lambda c^4}{8\pi G}. \tag{3.25}$$

This is a **fluid of constant density and negative pressure**, called 'dark energy' and acts to make the universe expand.

Predictions based off observations suggest that

- Dark energy is about 68% of $T^{\mu\nu}$
- Dark matter (matter we cannot see) is about 27% of $T^{\mu\nu}$

A static universe *is* possible if $\Lambda \neq 0$ but it's unstable.

Chapter 4

Gravitational waves

4.1 Linear GR

Small disturbances to vacuum spacetime (Minkowski) come in the form of a small perturbation $h_{\alpha\beta}$ where $|h_{\alpha\beta}| \ll 1$. Indeed,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x^\alpha) \tag{4.1}$$

We substitute this metric into the field equations and evaluate all terms, scalars and Christoffel symbols

$$\begin{aligned}\Gamma^\gamma_{\alpha\beta} &= \frac{1}{2}g^{\nu\alpha}\left(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu}\right)\\ &\simeq \frac{1}{2}\eta^{\nu\alpha}\left(h_{\nu\alpha,\beta} + h_{\nu\beta\alpha} - h_{\alpha\beta,\nu}\right).\end{aligned} \tag{4.2}$$

The Ricci tensor can be approximated to first order also:

$$\begin{aligned}R_{\mu\nu} =&\partial_\alpha\Gamma^\alpha_{\mu\nu} - \partial_\mu\Gamma^\alpha_{\alpha\nu} + \Gamma^\beta_{\mu\nu}\Gamma^\alpha_{\alpha\beta} - \Gamma^\beta_{\mu\alpha}\Gamma^\alpha_{\beta\nu}\\ =&\frac{1}{2}\eta^{\alpha\beta}\left(\partial_\alpha\partial_\mu h_{\beta\nu} + \partial_\alpha\partial_\nu h_{\mu\beta} - \partial_\alpha\partial_\beta h_{\mu\nu}\right)\\ &-\frac{1}{2}\eta^{\alpha\beta}\left(\partial_\mu\partial_\alpha h_{\beta\nu} + \partial_\mu\partial_\nu h_{\alpha\beta} - \partial_\mu\partial_\beta h_{\alpha\nu}\right) + O(h^2)\\ =&\frac{1}{2}\eta^{\alpha\beta}\left(-\partial_\alpha\partial_\beta h_{\mu\nu} + \partial_\mu\partial_\beta h_{\alpha\nu} + \partial_\alpha\partial_\nu h_{\mu\beta} - \partial_\mu\partial_\nu h_{\alpha\beta}\right) + O(h^2)\end{aligned} \tag{4.3}$$

$$R_{\mu\nu} \cong \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\alpha\nu;\mu} + O\left(h^2\right) \tag{4.4}$$

It depends on second derivatives of the perturbation $h_{\mu\nu}$ reflecting the fact that curvature is associated to second derivatives. It also means that constant and linear terms in $h_{\mu\nu}$ carry no physical significance; they can be removed by coordinate transformations.

In vacuum, $R_{\mu\nu}$ are the field equations.

$$\begin{aligned}&-\Box h_{\mu\nu} + \partial_\mu V_\nu + \partial_\nu V_\mu = 0\\ &\Box = \eta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\\ &v_\nu = h^\alpha_{\nu,\alpha} - \frac{1}{2}h_{,\nu},\end{aligned} \tag{4.5}$$

where $h = h^\alpha_\alpha$

4.2 Gravitational Waves (GWs)

We can always make a coordinate transformation $x^\alpha \to x^\alpha + \varepsilon^\alpha$ for $|\varepsilon^\alpha|$ small. Now, since $R_{\mu\nu} = 0$ is a tensor equation, it is valid under all coordinates.

Equations involving $h_{\mu\nu}$ are not valid tensor equations because of the dependence on ε^α. To see how this affects us, we first make a change of variables

Definition 4.2.1. The **trace-reversed perturbation** $\bar{h}_{\mu\nu}$ is defined as

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \tag{4.6}$$

We choose ε^α such that

$$\begin{aligned} \partial_\nu \bar{h}^{\mu\nu} &= 0 \\ \Box \bar{h}_{\mu\nu} &= 0 \end{aligned} \tag{4.7}$$

where of course,

$$\bar{h}^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}\bar{h}_{\alpha\beta},$$

i.e., $\bar{h}_{\mu\nu}$ satisfies the wave equation in Minkowski space.

The goal of this section is to find an expression for $\bar{h}_{\mu\nu}$ and what it could physically correspond to - i.e. sources of gravitational waves. The following derivation is non-examinable and is taken from Dr. Gareth's Alexander's notes with modification.

Theorem 4.2.1.

$$\bar{h}_{ij} = -\frac{2G}{c^4 r}\frac{\partial^2}{\partial t^2} I_{ij}, \tag{4.8}$$

where I_{ij} is the moment of inertia defined as

$$I_{ij} = \int \rho x^i x^j dV, \tag{4.9}$$

where

- *dV is volume of compact source*
- *r is the distance from source to observer*
- *ρ is the mass density of the source at the retarded time $t_r = t - r/c$*

Proof. Eq. (4.7) is an inhomogeneous (tensor) wave equation with the stress-energy-momentum tensor playing the role of the source. Minkowski space has $\mathbb{R}^4$ translational symmetry which allows a Fourier transform to be done (both sides are also integrable since they are continuous functions)

$$\begin{aligned} \bar{h}_{\mu\nu}(t, \mathbf{x}) &= \frac{-16\pi G}{c^4} \int \frac{\mathrm{e}^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}}{(2\pi)^4} \frac{\tilde{T}_{\mu\nu}}{(\omega/c)^2 - k^2} d\omega d^3k \\ &= \frac{-16\pi G}{c^4} \int G^{\text{ret}}(\mathbf{x} - \mathbf{x}', t - t')\, T_{\mu\nu}(\mathbf{x}', t')\, dt' d^3x' \end{aligned} \tag{4.10}$$

where $\tilde{T}_{\mu\nu}$ is the Fourier transform of the stress-energy-momentum tensor, $k = |\mathbf{k}|$, $G^{\text{ret}}(\mathbf{x}, t)$ is the retarded Green function for the wave operator, and we have made use of the convolution theorem. The appearance of the retarded Green function comes from physical considerations of the relevant boundary conditions; the matter content is a source for the gravitational radiation and a compact source should produce outgoing waves rather than ingoing waves. The gravitational disturbance produced by a compact source can therefore be given as

$$\begin{aligned}\bar{h}_{\mu\nu} &= \frac{-16\pi G}{c^4}\int \frac{-1}{4\pi\left|\mathbf{x}-\mathbf{x}'\right|}\delta\left((t-t')-\left|\mathbf{x}-\mathbf{x}'\right|/c\right)T_{\mu\nu}\left(\mathbf{x}',t'\right)dt'd^3x' \\ &= \frac{16\pi G}{c^4}\int \frac{T_{\mu\nu}\left(\mathbf{x}',t-\left|\mathbf{x}-\mathbf{x}'\right|/c\right)}{4\pi\left|\mathbf{x}-\mathbf{x}'\right|}d^3x'.\end{aligned} \tag{4.11}$$

The disturbance has to come from somewhere, and we attribute this to so-called gravitational waves. These are transverse waves[1] and therefore associated with the spatial components of the perturbed metric $\bar{h}_{ij}$ where $i,j=1,2,3$. Focusing on these components and looking at far-field solutions $|\mathbf{x}| \gg |\mathbf{x}'|$:

$$\bar{h}_{ij} = \frac{4G}{c^4|\mathbf{x}|}\int_{\text{source}} T_{ij}\left(\mathbf{x}',t-|\mathbf{x}|/c\right)d^3x'. \tag{4.12}$$

As described in Section 1.6, the energy-momentum tensor obeys a continuity equation $\partial^\mu T_{\mu\nu}=0$. We will use this to show the source of GWs is characteristic of a **mass quadrupole**. You have seen quadrupoles before in physics:

- electric quadrupole field (4 electric charges in a square)
- quadrupole magnetic field (2 spaced coils with current in the same direction)

We can therefore integrate T_{ij}, abusing symmetry $T_{\mu\nu}=T_{\nu\mu}$:

$$\begin{aligned}\int T_{ij}d^3x &= \frac{-1}{2c}\frac{\partial}{\partial t}\int\left[x^jT_{i0}+x^iT_{j0}\right]d^3x \\ &= \frac{-1}{2c}\frac{\partial}{\partial t}\int\left[x^j\left(\partial_kx^i\right)T_{0k}+x^i\left(\partial_kx^j\right)T_{0k}\right]d^3x \\ &= \frac{1}{2c}\frac{\partial}{\partial t}\int x^ix^j\partial_kT_{0k}d^3x \\ &= \frac{1}{2c^2}\frac{\partial^2}{\partial t^2}\int x^ix^jT_{00}d^3x\end{aligned} \tag{4.13}$$

But what is T_{00}? It is nothing more than ρc^2! Moreover, the integral is pretty much the expression for the moment of inertia. We arrive at the desired formula.

$$\bar{h}_{ij}(\mathbf{x},t) = \frac{2G}{c^6|\mathbf{x}|}\frac{\partial^2}{\partial t^2}\int_{\text{source}} x'^ix'^jT_{00}\left(\mathbf{x}',t-|\mathbf{x}|/c\right)d^3x' \tag{4.14}$$

To convert the notation, simply recast $d^3x' \to dV$, $|\mathbf{x}| \to r$ and replace T_{00} with ρc^2 (remembering to evaluate at the retarded time) and we are done. □

From this equation for $\bar{h}_{ij}$, we note a couple things

- Waves travel at c
- No time components at all in Lorenz gauge - only space perturbations
- Traceless and transverse coordinates for $\bar{h}_{ij}$

[1]Physical solutions of the field equations end up producing transverse solutions.

4.3 Properties of gravitational waves

We suppose we have propagation of $h_{\mu\nu}$ in the z direction

$$h_{\mu\nu}(z,t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a_{11} & a_{12} & 0 \\ 0 & a_{12} & -a_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \exp\left[i\omega t - i\omega z/c\right] \tag{4.15}$$

This perturbation tells us some remarkable properties

- No time-dependence other than in the waveform $\exp\left[i\omega t - i\omega z/c\right]$
- Only 2 polarisations: a_{11}, a_{12}
- Transverse waves propagate at c.

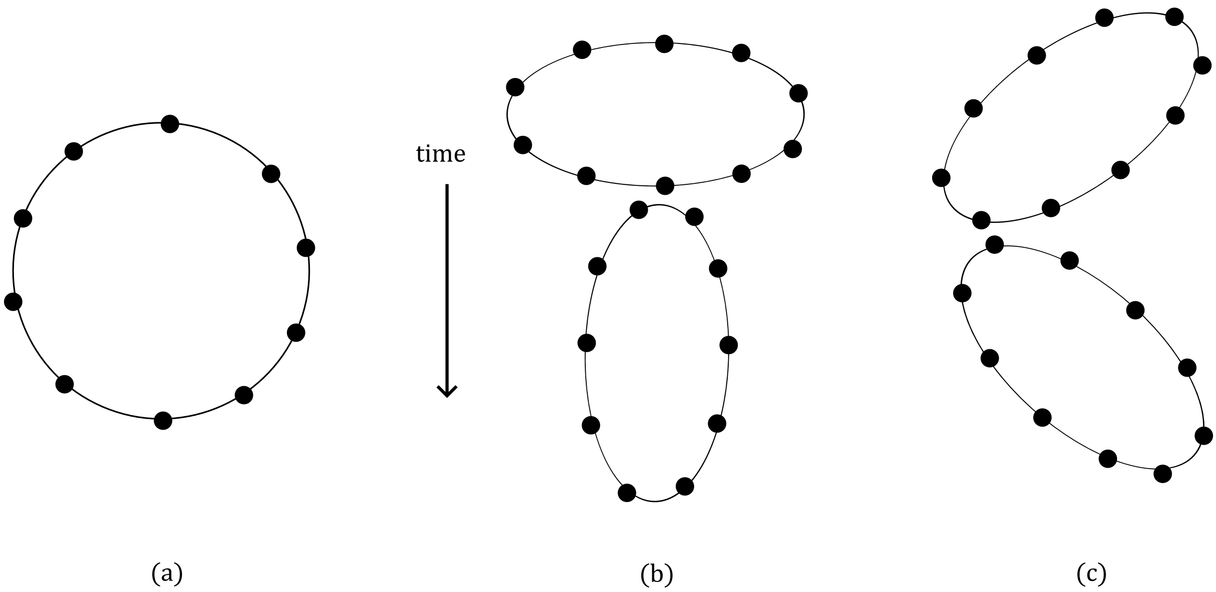

Figure 4.1: Identical test masses in a ring with (a) no GW, (b) +-polarisation, and (c) × polarisation.

Consider a circle of identical test masses as in Fig. 4.1. The 2 polarisations are called the + polarisation and × polarisation. In the + polarisation, the ring oscillates compression and expansion along x, y as shown in Fig. 4.1(b). The metric for this is given by

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu + h\cos(\omega(t - z/c))\left(dx^2 - dy^2\right) \tag{4.16}$$

In the × polarisation, these ellipses are oriented diagonally as in Fig. 4.1(c). To mathematically describe these GWs, we can consider rotating the + polarisation by some angle ϕ. Therefore, ds^2 is similar but

$$dx^2 - dy^2 \to \cos 2\phi\left[dx^2 - dy^2\right] + 2\sin 2\phi dxdy$$

so the final metric is

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu + h\cos(\omega(t - z/c))\left(\cos 2\phi\left[dx^2 - dy^2\right] + 2\sin 2\phi dxdy\right) \tag{4.17}$$

4.4 Generating GWs

Since $\bar{h}_{\mu\nu}$depends on I_{ij}, generating noticeable GWs requires systems with a large moment of inertia. Individual spinning stars, or other objects that are axisymmetric, are therefore not good sources. However, binary star systems are pretty good. Consider 2 identical stars each of mass M, initially placed a distance a away, rotating with frequency Ω about the centre-of-mass. The axis line is oriented parallel to x and the system is placed in the $x-y$ plane.

The mass density is then

$$\rho = M\delta\left(x - x_1\right) + M\delta\left(x - x_2\right) \tag{4.18}$$

when the y-coordinates are 0. The x positions of the masses obey

$$x_1 = \frac{a}{2}\cos(\Omega t) \quad x_2 = -\frac{a}{2}\cos\Omega t \tag{4.19}$$

By evaluating the $\bar{h}_{ii}$ components, the amplitude (prefactor) can be found. We first calculate I_{xx}:

$$I_{xx} = \int x^2 p(x,t)dx = 2M\left(\frac{a}{2}\cos\Omega t\right)^2 = \frac{1}{4}Ma^2(1+\cos(2\Omega t)) \tag{4.20}$$

so

$$h_{xx} = \frac{-2GM\Omega^2a^2}{c^4r}\cos(2\Omega t).$$

Namely, **GWs have a frequency double the source**.

Using Kepler's third law, $\Omega^2a^3 = 2GM$

$$\frac{2GM\Omega^2a^2}{c^4r} = \frac{\left(2GM/c^2\right)^2}{ar} = \frac{(2\mu)^2}{ar} = \frac{R_\mathrm{s}^2}{ar} \tag{4.21}$$

The amplitude is tiny; it is the product of the Schwarzschild radii for the two stars divided by their separation and the distance they are from the point of observation. Not only is the Schwarzschild radius small on astronomical scales, but the distance to the source is truly astronomical. This makes GWs very hard to detect.

4.5 Detecting GWs

Due to the binary star system, the metric is perturbed as

$$ds^2 = (g_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu \tag{4.22}$$

The ttypical distance will be measured as

$$ds = \left(1 + \frac{1}{2}h\right)L, \tag{4.23}$$

where L is the unperturbed length. Because of Minkowski space, the fractional change is the *measured* proper length.

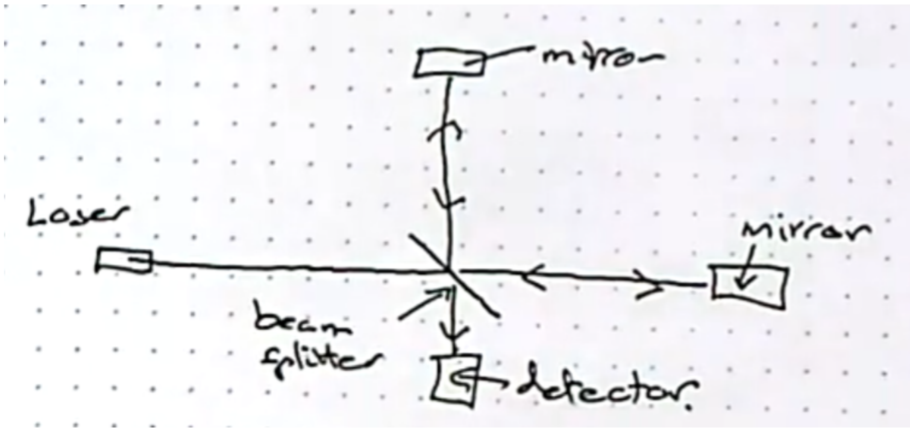

Figure 4.2: Michelson interferometer sketch from Prof. Tony Arber's lectures.

4.5.1 LIGO and VIRGO detectors

These are long-arm (LIGO about 4 km, VIRGO about 3 km) Michelson interferometers, a schematic is shown in Fig. 4.2. We have a laser system (a laser pointer is *not* enough) which is incident on a beam splitter, producing a 50/50 split down 2 paths which are of identical lengths. The light then bounces back and forth say 100+ times and then they recombine.

The paths are configured such that once the light rays return to the detector, if there is **no GWs** then they **completely destructively interfere** - i.e. you detect nothing. Then, if there is a perturbation, the lengths light travels is different between the paths and a the recombination no longer completely destructively interferes.

1. Hypothetically you could replace this setup with a cavity interferometer (Fabry-Perot, optical cavity), but that's just not feasible because how well do you think multiple-kilometre mirrors are gonna be?

2. The existence of polarisation states: suppose you have the +-polarisation which are oriented vertically and horizontally. This means there is contraction in one axis and expansion in another, since the paths are perpendicular, this will be detected.

There are 2 LIGO detectors across the US, allowing you to correlate the signal. This is necessary, because these detectors will be sensitive to *anything* that vibrates, including earthquakes, construction, even the air hitting the mirrors. Hence the 2 LIGO detectors are also oriented differently to analyse any polarisations and remove noise etc. Triangulation (i.e., locating the source of the GW) is done with the VIRGO detector in Italy.

The detectors are sensitive to frequencies (LIGO) between 400 to 10,000 Hz. For lower frequency, this is typical sound waves. However, seismic activity, lightning etc. distort the accuracy of low-frequency vibrations.

LIGO is sensitive to GWs down to $h \sim 10^{-21}$, so a change in arm lengths of about 10^{-18} metres.

Using Kepler's third law, $\Omega^2 a^3 = 2GM$, to increase the amplitude of the signal, we need smaller a (separation). For a binary system, this small separation typically occurs in the **inspiral stage**, when the 2 stars are spiralling towards each other and about to collide. This corresponds to the final 10-20 seconds!

Some examples of what we can detect:

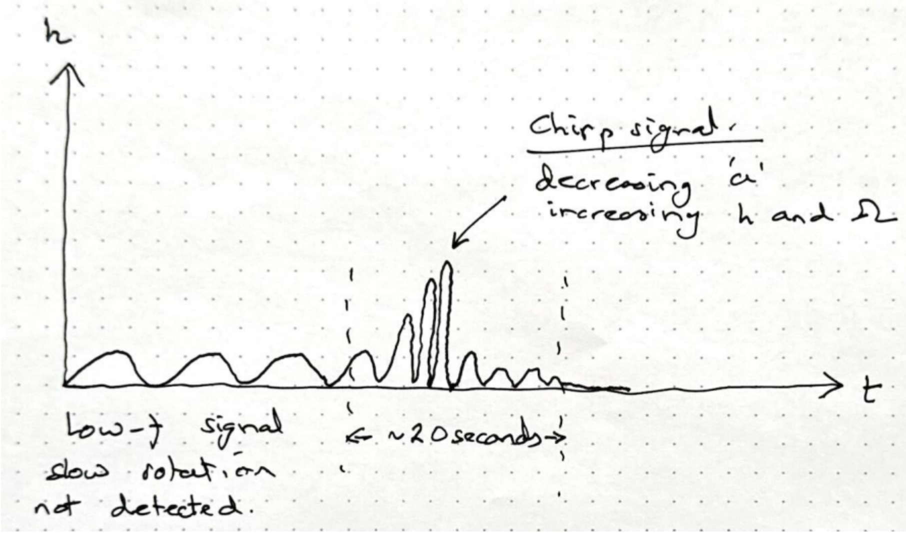

Figure 4.3: Typical chirp signal sketch. Screenshot from Prof. Tony Arber's lectures.

- 20-30 solar mass Kerr black hole mergers. $r \sim 400$ Mpc - extra-galactic source
- 1-2 solar mass neutron stars at about $d \sim 30$ Mpc - extra-galactic
- a decreases due to a loss of energy to GWs. This leads to an increase of h and Ω and the chirp signal is calculated numerically

4.5.2 Chirp mass

A typical chirp signal looks like Fig. 4.3 for a rotating binary system. You should be able to recall this type of graph for the exam.

Using Kepler's law, you can determine the mass of the binary system. Relaxing our assumption of identical stars in the binary, we label them with masses m_1, m_2

$$\Omega^2 = \frac{(m_1 + m_2)c^2}{a^3} \tag{4.24}$$

Differentiate both sides w.r.t time

$$\begin{aligned} 2\Omega\dot{\Omega} &= -\frac{3\,(m_1+m_2)\,c^2}{a^4}\dot{a}, \\ &= \frac{192c^3 m_1 m_2\,(m_1+m_2)^2}{5a^7}, \\ &= \frac{192}{5}c^{-5/3}\frac{m_1 m_2}{(m_1+m_2)^{1/3}}\Omega^{14/3}. \end{aligned} \tag{4.25}$$

This can rearranged into an expression for the so-called **chirp mass**, the effective mass of a binary system:

$$\mathcal{M} = \left(\frac{m_1^3 m_2^3}{m_1+m_2}\right)^{1/5} = c\left(\frac{5}{96}\right)^{3/5}\Omega^{-11/5}\dot{\Omega}^{3/5} \tag{4.26}$$

where the first equality is derived from a so-called 'post-Newtonian expansion' and is non-examinable.

Chapter 5

Exam questions

Some answers to past exam questions. Words highlighted in **bold** are keywords that you may want to look for during the exam.

5.1 Geodesics

5.1.1 2019 Q2a

> **EXAMPLE 1.**
>
> Describe briefly the significance of geodesics in GR and contrast this with the situation in Newton's theory of gravity. No calculations are required [6].

Proof. We have to do 2 make sure we do two things in our answer. For ever y point we make about geodesics, we *must* compare with Newton to get the marks.

- Motion of free massive particles is along time-like geodesics of space-time
- Light travels along null geodesics
- This behaviour is seen in planetary orbital motion, gravitational lensing, Shapiro delay, redshift
- Newton's theory: all massive objects experience a gravitational force from all other massive objects
- Free particles, experiencing no force, travel along geodesics - in flat Euclidean space, these are just straight lines
- Light also travels along straight line geodesics in Newton, and do not experience any relativistic effects.

□

5.2 The Field Equations

5.2.1 2019 Q1c

Example 2.

The source of gravity is the stress-energy-momentum tensor, so that the full Einstein equations take the form

$$R_{\mu v} - \frac{1}{2} R g_{\mu v} = \kappa T_{\mu v}$$

The constant κ can be determined from consideration of the **Newtonian limit**, which you should do as follows: (i) Use the **geodesic equation**, $\frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^v}{d\tau} = 0$, where $\tau = c\times$ proper time, to show that

$$\Gamma^i_{00} \simeq \frac{1}{c^2} \partial_i \phi,$$

in the Newtonian limit, where ϕ is the Newtonian potential. $\{\mathbf{3}\}$
(ii) Using the previous result and the general formula given in the rubric, show that

$$R_{00} \simeq \frac{1}{c^2} \partial_i \partial_i \phi$$

(iii) Show that $R = -\kappa T$, where $T = g^{\mu\nu} T_{\mu\nu}$.
(iv) Finally, taking $T_{\mu\nu}$ to be dominated by the energy density, i.e. $T_{00} \simeq \rho c^2$ and all other components negligible, determine the value of κ.

This is considered bookwork since this was explicitly covered in lectures or the problem sheet.

Proof. (i) We consider $\mu = \nu = 0$. Indeed then, $\dot{x}^0 = \gamma c = \gamma$ because in the Newtonian limit, objects move slowly so $c \gg v$ [**1**]. Notice then we get

$$\frac{1}{c^2} \frac{d^2 x^\alpha}{d\tau^2} = \Gamma^\alpha_{00} \tag{5.1}$$

LHS is a factor proportional to acceleration. Newton tells us that this must be proportional to the gradient of a potential $-\nabla\phi = -\partial_i \phi$ and so we get the desired relationship [**2**].

(ii) This is direct application of Eq. (1.107). In the weak field limit, second derivatives (product of Christoffel symbols) can be neglected and Newtonian gravity is time-independent. Considering R_{00} we get

$$R_{00} \simeq \partial_i \Gamma^i_{00} \simeq \frac{1}{c^2} \partial_i \partial_i \phi \tag{5.2}$$

(iii) Contract with the fully-contravariant metric tensor $g^{\mu\nu}$ on both sides. The question hints you on this by telling you what the scalar T is defined as.

$$g^{\mu v} R_{\mu v} - \frac{1}{2} R g^{\mu v} g_{\mu v} = \kappa g^{\mu v} T_{\mu v} \tag{5.3}$$

Recall a property of product of metric tensors, Eq. (1.3.2) which simplifies the product to be 4. Rearrange to get R by itself and we are done.

(iv) We need only consider the 00-component as in lectures. This is identical to Section 1.11.2. □

5.2.2 2019 Q1d

Example 3.

In the space-time region external to the Sun, the stress-energy-momentum tensor is zero and the metric satisfies the vacuum Einstein equations. Show that this means the space-time geometry external to the Sun satisfies $R_{\mu\nu} = 0$, i.e. the Ricci curvature vanishes identically. Comment on whether or not the space-time is curved [4].

Proof. External to the Sun, energy tensor is 0. Hence, we again contract with the contravariant metric tensor

$$0 = g^{\mu v} R_{\mu v} - \frac{1}{2} R g^{\mu v} g_{\mu v} = R - \frac{1}{2} R(4) = -R, \tag{5.4}$$

So the Ricci scalar vanishes. However the Ricci scalar is a sum of metric components and Ricci tensor. Indeed even for empty, flat space, the metric is non-zero so the Ricci tensor must vanish everywhere.

Recall that

- $R_{\mu\nu\alpha\beta} = 0 \implies$ space is not curved and flat everywhere
- $R_{\mu\nu} = 0, R_{\mu\nu\alpha\beta} \neq 0 \implies$ space is **not necessarily flat** - space can still curve in the intuitive sense - **Ricci-flat**
- Both tensors non-zero, definitely not flat in any way

So space time us curved (for extra brownie points, you might note the planets travel on time-like geodesics of the metric but they aren't geodesics in flat Minkowksi space). □

5.3 Time dilation

5.3.1 2019 Q2c-e

EXAMPLE 4.

Derivation of the Shapiro delay

c) Consider a radar (light) signal passing close to a massive gravitational body, described by the Schwarzschild metric with the trajectory of the signal lying in the equatorial plane ($\theta = \pi/2$). Along the trajectory, the azimuthal coordinate is related to the time coordinate by

$$r^2 d\phi = \frac{\ell}{\gamma}\left(1 - \frac{2\mu}{r}\right) cdt$$

where ℓ and γ are the two conserved first integrals of the motion. Show that the trajectory of the radar signal satisfies

$$-\left(1 - \frac{2\mu}{r}\right)\left[1 - \frac{(\ell/\gamma)^2}{r^2}\left(1 - \frac{2\mu}{r}\right)\right] c^2 dt^2 + \frac{1}{1 - \frac{2\mu}{r}} dr^2 = 0.$$

d) Use the **distance of closest approach**, $r = b$, to determine ℓ/γ and consequently show that

$$cdt = \frac{1}{1 - \frac{2\mu}{r}}\left[1 - \frac{b^2}{r^2}\frac{\left(1 - \frac{2\mu}{r}\right)}{\left(1 - \frac{2\mu}{b}\right)}\right]^{-1/2} dr.$$

e) Working to **first order** in μ, show that the travel time for the radar signal between the distance of closest approach, $r = b$, and a position $r = R$ is given by

$$c \times \text{ travel time } = \sqrt{R^2 - b^2} + 2\mu\left\{\ln\left[\frac{R}{b} + \sqrt{\frac{R^2}{b^2} - 1}\right] + \frac{1}{2}\sqrt{\frac{R-b}{R+b}}\right\}.$$

What interpretation can be given to the two terms in this expression? {7}
[You may quote the standard integrals $\int \frac{dx}{\sqrt{x^2-1}} = \ln\left(x + \sqrt{x^2-1}\right)$ and $\int \frac{dx}{(x+1)\sqrt{x^2-1}} = \sqrt{\frac{x-1}{x+1}}$.]

This question is more about algebra and maths than any actual physics, especially the big 7 marker, where you will *not* get many marks if you don't spot how to rearrange it.

Proof. (c) Radar is light, and it travels along null geodesics $ds^2 = 0$, so using the full Schwarzchild metric gives

$$ds^2 = -\left(1 - \frac{2\mu}{r}\right) c^2 dt^2 + \frac{1}{1 - \frac{2\mu}{r}} dr^2 + r^2 d\phi^2 = 0. \tag{5.5}$$

for which you can eliminate $d\phi^2$ directly.

(d) Closest approach means $dr = 0$. Rearranging the result of (c) with $r = b$ gives

$$(\ell/\gamma)^2 = b^2\left(1 - \frac{2\mu}{r}\right)^{-1} \tag{5.6}$$

Now you substitute this back into (c) and square root both sides.

(e) The painful part. We first have to manipulate the square root part.

$$
\begin{aligned}
1-\frac{b^2}{r^2}\frac{\left(1-\frac{2\mu}{r}\right)}{\left(1-\frac{2\mu}{b}\right)} &= 1-\frac{b^2}{r^2}-\frac{b^2}{r^2}\frac{\left(\frac{2\mu}{b}-\frac{2\mu}{r}\right)}{\left(1-\frac{2\mu}{b}\right)}, \\
&= \left(1-\frac{b^2}{r^2}\right)\left[1-\frac{2\mu b}{r^2}\frac{1}{1-\frac{b^2}{r^2}}\frac{\left(1-\frac{b}{r}\right)}{\left(1-\frac{2\mu}{b}\right)}\right], \\
&= \left(1-\frac{b^2}{r^2}\right)\left[1-\frac{2\mu}{r}\frac{1}{\frac{r}{b}+1}\frac{1}{1-\frac{2\mu}{b}}\right].
\end{aligned}
\tag{5.7}
$$

where to go from the first to second line, we add and subtract a 1, then factor out $(1-b^2/r^2)$, and from the second to the third line, multiply put the second term in suare brackets and simplify.

Now, the LHS is cdt, so integrating gives ct, where t is the travel time. Then the RHS is integrated between $[b, R]$:

$$
\begin{aligned}
ct &= \int_{r=b}^{R}\left(1-\frac{b^2}{r^2}\right)^{-1/2}\left\{1+\frac{2\mu}{r}+\frac{1}{2}\frac{2\mu}{r}\frac{1}{\frac{r}{b}+1}\right\}dr, \\
&= \int_{r=b}^{R}\frac{r}{\sqrt{r^2-b^2}}dr+2\mu\int_{r=b}^{R}\left\{\frac{1}{\sqrt{r^2-b^2}}+\frac{1}{2}\frac{1}{\frac{r}{b}+1}\frac{1}{\sqrt{r^2-b^2}}\right\}dr, \\
&= \sqrt{R^2-b^2}+2\mu\left\{\ln\left(\frac{R}{b}+\sqrt{\frac{R^2}{b^2}-1}\right)+\frac{1}{2}\sqrt{\frac{R-b}{R+b}}\right\},
\end{aligned}
\tag{5.8}
$$

To get the curly brackets in the first line, expand the $1/(1-2\mu/b)$ in the square brackets to first order. To get from the first line to the second line, expand the brackets in to square root form and distribute. then the third line uses the standard integrals. □

5.4 FLRW

5.4.1 2019 Q4e

Example 5.

e) Consider a light signal travelling through a FLRW space-time that was emitted from a nearby galaxy at $\chi = \chi_{\text{emit}}$ at a time $t = t_{\text{emit}}$ and is observed at our own galaxy at $\chi = 0$ at the present time $t = t_{\text{now}}$. The signal may be assumed to travel along a 'radial' line with θ and ϕ fixed.

(i) Show that

$$\chi_{\text{emit}} = \int_{t_{\text{emit}}}^{t_{\text{now}}} \frac{c\,dt}{a(t)}$$

(ii) Considering this as a relation between the observation time, t_{now}, and the emission time, i.e. $t_{\text{now}} = t_{\text{now}}(t_{\text{emit}})$, show that

$$\frac{dt_{\text{now}}}{dt_{\text{emit}}} = \frac{a(t_{\text{now}})}{a(t_{\text{emit}})} \approx 1 + \left.\frac{\dot a}{a}\right|_{t_{\text{now}}} (t_{\text{now}} - t_{\text{emit}})$$

(iii) If the nearby galaxy from which the signal is emitted is considered to be receding from us with line-of-sight velocity v, the frequency of the emitted light, ω_{emit}, will be Doppler shifted relative to that which we observe, ω_{now}, so that

$$\omega_{\text{emit}} = \omega_{\text{now}} \left(1 + \frac{v}{c}\right)$$

Show that this together with the previous result implies that the recession velocity obeys Hubble's law, $v = H_{\text{now}}\, d$, where d is the (current) distance to the nearby galaxy.

Proof. (i) Integrate along radial null geodesic, so $d\theta = d\phi = 0$ in FLRW metric, so

$$ds^2 = 0 = -c^2 dt^2 + a^2(t) d\chi^2 \iff \frac{c}{a(t)} dt = d\chi \tag{5.9}$$

Integrate the RHs between $\chi \in [0, \chi_{\text{emit}}]$ and the LHS between $[t_{\text{emit}}, t_{\text{now}}]$ and we get the desired integral.

(ii) We differentiate under the integral sign, the expression in (i). This is done by the Leibniz formula

$$\frac{\partial}{\partial z} \int_{a(z)}^{b(z)} f(x,z)dx = \int_{a(z)}^{b(z)} \frac{\partial f}{\partial z} dx + f(b(z),z)\frac{\partial b}{\partial z} - f(a(z),z)\frac{\partial a}{\partial z} \tag{5.10}$$

where in our case, $z = t_{\text{emit}}, b(z) = b(t_{\text{emit}}) = t_{\text{now}}(t_{\text{emit}}), a(z) = a(t_{\text{emit}}) = t_{\text{emit}}$. Differentiating then, we get

$$0 = \frac{c}{a(t_{\text{now}})} \frac{dt_{\text{now}}}{dt_{\text{emit}}} - \frac{c}{a(t_{\text{emit}})}, \quad \Longrightarrow \quad \frac{dt_{\text{now}}}{dt_{\text{emit}}} = \frac{a(t_{\text{now}})}{a(t_{\text{emit}})}. \tag{5.11}$$

Expanding $a(t_{\text{emit}})$ about the current time as a first order Taylor series:

$$\frac{dt_{\text{now}}}{dt_{\text{emit}}} \approx \frac{a(t_{\text{now}})}{a(t_{\text{now}}) - \dot a|_{t_{\text{now}}} (t_{\text{now}} - t_{\text{emit}})} \approx 1 + \left.\frac{\dot a}{a}\right|_{t_{\text{now}}} (t_{\text{now}} - t_{\text{emit}}). \tag{5.12}$$

(iii) Conceptually, the Doppler shift relates the ratio of time intervals to a frequency shift. Noting then the numerator and denominator of $dt_{\text{now}}/dt_{\text{emit}}$ are themselves time intervals, they are each inversely proportional to angular frequency

$$\frac{dt_{\text{now}}}{dt_{\text{emit}}} = \frac{\omega_{\text{emit}}}{\omega_{\text{now}}} \approx 1 + \frac{v}{c}. \tag{5.13}$$

Now, (ii) contains an expression for the derivative in terms of $a, \dot{a}$. We equate both sides ad assume $(t_{\text{now}} - t_{\text{emit}}) = d/c$. Then the recession velocitiy and thus Hubble's law is

$$v = \left.\frac{\dot{a}}{a}\right|_{t_{\text{now}}} d = H_{\text{now}}\, d \tag{5.14}$$

□

5.5 Gravitational waves

5.5.1 2019 Q3a, 2024 Q3a

Example 6.

Summarise the LIGO experiment and its successful detection of gravitational waves. Your answer should cover: the nature of the experiment; the nature of the signal detected; some details of the inferred source(s) [8].

Proof. It's enough to state these bullet points

- LIGO experiment consists of **two long-arm Michelson interferometers** placed on **opposite sides of the US**.
- They are **sensitive to strains around** 10^{-21} **m** and the arms are **multi-kilometres long** to try to detect this
- Sensitive to frequencies between **400 and 10,000 Hz**
- Signals **correlated** between the two LIGO detectors to remove noise.
- **Triangulation of the GW done with the VIRGO** detector in Italy, to try to find the source of the GW
- Detected signal is a **chirp** - amplitude and frequency *increase* as bodies approach to merge. Then after merging, a sudden decrease
- Source nature is inferred by numerically solving the field equations
- Successfully inferred sources of GWs can be 2 Kerr black holes or a binary neutron star system.

□